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THE UNIVERSITY OF ALBERTA

OPTIMAL DISCRETE OBSERVERS

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Optimal Discrete Observers submitted by Chong G. Kim in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The problem of designing an observer which is both minimum-cost and minimum-time is considered in this thesis. By cost is meant a quadratic functional of the estimation error.

For minimum-time observers the procedure involves a selection of a minimal nilpotent matrix by using a suitable canonical representation for the plant, so that the estimation errors vanish in a minimum number of steps. For minimum-cost observers, the procedure is based on a minimization of the cost due to the estimation error. The dependence of the performance measure on the initial conditions of the unmeasurable state variables is eliminated by averaging the performance index.

The choice of a particular canonical representation for the system leads to a simple procedure for observers which simultaneously minimize the performance measure on the average cost and the time required to yield errorless state variables. Computer simulation results showing the performance of the various observers are presented.

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CHAPTER 1

INTRODUCTION

1.1 General Introduction

Systems are designed and built so as to perform certain defined functions such as, for example, pole-assignment or regulation. In order for the systems to carry out these functions, we adopt, in general, the feedback technique in which knowledge of the entire state of a system is required. However, there are, in practice, many situations in which some of the state variables are not available for direct measurement. In these situations, the control law requiring all the state variables cannot be implemented without additional hardware.

Either an approach in which the control variables are generated directly from the available outputs of the system must be used [1]* or estimates of the unmeasurable state variables must be made which can be substituted into the control law. In general, it has been found that the latter approach is vastly simpler than a direct method mentioned earlier [2].

Two dynamic systems have been developed for estimating the state variables of a system: Kalman filter for a stochastic system and Luenberger observer for

---

* Numbers in [ ] denote references listed at the end of this thesis.

a deterministic system. A Kalman filter estimates the entire state vector such that the variance of the estimation error is minimized in a noisy environment. The parameters of a Kalman filter are thus directly determined by the knowledge of the statistical characteristics of the environmental noise. On the other hand, the designer has some freedom in determining the dynamics of a Luenberger observer which estimates a linear transformation of the unmeasurable states of the given system. This freedom, in turn, poses another attractive problem in observer design procedure, and as a consequence it has received much attention in recent years.

In this thesis, we discuss several useful design methods that have been developed by others and present a new method of observer design. The resulting observer possesses attractive characteristics. We shall restrict the discussion to linear time invariant discrete systems.

1.2 Theory of Observation

i) Basic Theory

Consider the completely observable linear time-invariant discrete system governed by:

$$x(k+1) = Ax(k) + Bu(k); \qquad y(k) = Cx(k) \qquad (1\text{-}1)$$

where

x is an $n \times 1$ state vector,

u is a $p \times 1$ control vector,

y is an $m \times 1$ output vector,

A is an $n \times n$ plant matrix,

B is an $n \times p$ distribution matrix,

C is an $m \times n$ output matrix

and without loss of generality, C is assumed to be of full rank, and the "observer" governed by:

$$z(k+1) = Fz(k) + Gy(k) + Hu(k) \tag{1-2}$$

which is driven by the inputs and outputs of the system (1-1) where the observer matrix F is a stable matrix (i.e. the eigenvalues are within the unit circle). Then we can write:

$$z(k+1) - Tx(k+1) = F[z(k) - Tx(k)] + [GC - TA + FT]x(k) + [H-TB]u(k) \tag{1-3}$$

and $$z(k+1) - Tx(k+1) = F[z(k) - Tx(k)] \tag{1-4}$$

if T is chosen such that:

$$TA - FT = GC \tag{1-5a}$$

$$H = TB \tag{1-5b}$$

Since F is a stable matrix the solution to the homogeneous eqn. (1-4) is:

$$z(k) - Tx(k) = F^k[z(0) - Tx(0)] \tag{1-6}$$

implying that $z(k) \to Tx(k)$ as $k \to \infty$

Thus the system (1-2) asymptotically estimates the linear transformation of the state vector of the system (1-1).

Furthermore, if $z(0) = Tx(0)$ then the observer [system (1-2)] yields an exact linear transformation of the state vector x at every step.

The condition for existence of the unique solution for T to eqn. (1-5) is that the matrices A and F do not have any eigenvalues in common [2],[3]. Thus all the eigenvalues of the matrix F must be chosen to be distinct from those of the matrix A. It should be noted that since the observer estimates a linear transformation of the state vector, the matrix T must have a rank large enough to enable the observer to recover all the unmeasurable state variables.

ii) Identity Observer

It would be obviously convenient to choose the transformation T relating the state $z(k)$ to the state $x(k)$ as the identity transformation I_n. Then eqn. (1-5)

becomes:

$$F = A - GC \tag{1-7a}$$

$$H = B \tag{1-7b}$$

Hence:

$$z(k+1) = (A - GC)z(k) + Gy(k) + Bu(k) \tag{1-8}$$

is an identity observer which is of the same order as the observed system (i.e. n).

While any nxm matrix G leads to an identity observer the dynamic response of the observing process is completely determined by the eigenvalues of the matrix A - GC. Since the eigenvalues of the matrix A - GC can be arbitrarily placed by a suitable choice of G if and only if the pair (A, C) is completely observable, we have the following result:

An identity observer having arbitrary dynamics can be constructed for a linear system if and only if the system is completely observable [2].

In practice, the eigenvalues of the observer are chosen to be closer to the origin than those of the observed system so that convergence is faster than other system effects. Notice that if the matrix G is chosen

such that the eigenvalues are all zero then the observer produces the exact state x in at most n steps.

iii) Reduced Order Observer

The identity observer though possessing an ample measure of simplicity also possesses a certain degree of redundancy. This comes from the fact that while the identity observer constructs an estimate of the entire state vector, part of the state vector as provided by the system outputs are available by direct measurement. This redundancy can be eliminated and an observer of lower order but still having arbitrary dynamics can be designed.

Since the output matrix C is of full rank m, the output vector y(k) itself provides m state variables through measurement. In order to estimate the remaining n-m state variables an observer of order n-m is constructed with the state z(k) which approximates Tx(k) for a suitable (n-m)xn matrix T. Then if the partitioned nxn matrix:

$$M = \begin{bmatrix} C \\ \hline T \end{bmatrix} \tag{1-9}$$

is nonsingular (i.e. invertible) an estimate $\hat{x}(k)$ of the entire state vector x(k) is determined by:

$$\hat{x}(k) = M^{-1} \begin{bmatrix} y(k) \\ \hline z(k) \end{bmatrix} \tag{1-10}$$

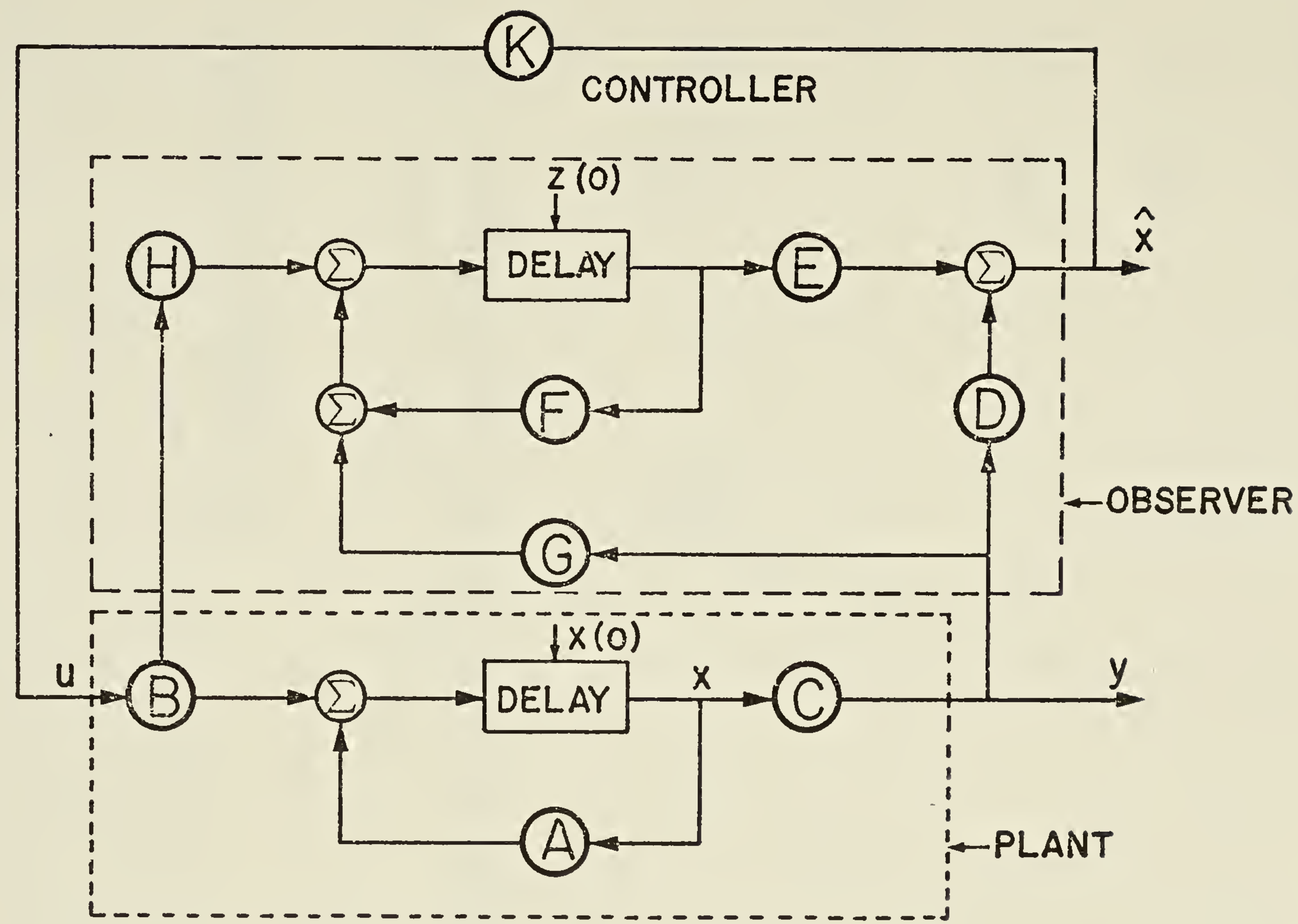

Figure 1: Reduced Order Observer-based Control System

Thus the matrix associated with the observer must have n-m independent rows which are also linearly independent of the rows of C.

The matrices F and G can be expressed in terms of the matrices T, D and E.

$$F = TAE \tag{1-11a}$$

$$G = TAD \tag{1-11b}$$

where

$$[\,\underbrace{D}_{n} \,\vdots\, \underbrace{E}_{n-m}\,] = M^{-1} = \begin{bmatrix} C \\ \hline T \end{bmatrix}^{-1} \tag{1-12}$$

The reduced order observer can be incorporated into the flow diagram as shown in Fig. 1.

1.3 Gopinath's Method [4]

It is shown in the following section that although any matrix T satisfying eqns. (1-5) and (1-9) leads to a reduced order observer, a particular choice of T leads to a simple and elegant design procedure.

Again the system under consideration is described by:

$$x(k+1) = Ax(k) + Bu(k); \qquad y(k) = Cx(k) \tag{1-13}$$

With no loss in generality, C can be assumed to be of the form:

$$C = [\, I_m \mid O \,] \tag{1-14}$$

by introducing a change of coordinates. It is convenient to partition the state vector x as:

$$x = \left[\begin{array}{c} y \\ \hline w \end{array}\right] \tag{1-15}$$

where w is the unmeasurable part of the state vector x and the matrices A and B as:

$$A = \begin{array}{c} \left.\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]\begin{array}{l} \}\, m \\ \}\, n-m \end{array}\right. \\ \quad m \quad\; n-m \end{array} \qquad B = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right]\begin{array}{l} \}\, m \\ \}\, n-m \end{array} \tag{1-16}$$

Then system is:

$$y(k+1) = A_{11}y(k) + A_{12}w(k) + B_1u(k) \tag{1-17a}$$

$$w(k+1) = A_{22}w(k) + A_{21}y(k) + B_2u(k) \tag{1-17b}$$

Since the system (1-17a) provides the measurement $A_{12}w(k)$ for the system (1-17b), which has w(k) as its n-m state vector and $A_{21}y(k) + B_2u(k)$ as its input, an

identity observer of order n-m is constructed for the system (1-17b) using this measurement. The theorem [see Appendix A] which says that if the pair (A, C) is completely observable, so is the pair (A_{22}, A_{12}) justifies the reconstruction of an estimate $\hat{w}(k)$ of w(k).

An identity observer for the system (1-17b) with the measurement:

$$A_{12}w(k) = y(k+1) - A_{11}y(k) - B_1u(k) \tag{1-18}$$

is given by:

$$\begin{aligned}\hat{w}(k+1) = &[A_{22} + LA_{12}]\hat{w}(k) - L[y(k+1) \\ &- A_{11}y(k) - B_1u(k)] + A_{21}y(k) + B_2u(k)\end{aligned} \tag{1-19}$$

where the elements of the (n-m)xm matrix L are the free design parameters which should be suitably chosen subject to the restriction that the matrix $A_{22} + LA_{12}$ is stable.

If z(k) is defined as $\hat{w}(k) + Ly(k)$, the requirement for measuring the delayed y(k) (i.e. y(k+1)) can be avoided:

$$\begin{aligned}z(k+1) = &(A_{22} + LA_{12})z(k) - (A_{22} + LA_{12})Ly(k) \\ &+ (A_{21} + LA_{11})y(k) + (B_2 + LB_1)u(k) \\ &= Fz(k) + Gy(k) + Hu(k)\end{aligned} \tag{1-20}$$

Eqn. (1-20) exclusively defines the matrices:

$$F = A_{22} + LA_{12} \tag{1-21a}$$

$$G = -FL + A_{21} + LA_{11} \tag{1-21b}$$

$$H = B_2 + LB_1 \tag{1-21c}$$

The observer state z(k) asymptotically estimates Tx(k), a linear transformation of the observed system vector, where $T = [\, L \,\vdots\, I_{n-m}]$, that is

$$z(k) = T\hat{x}(k) = Ly(k) + \hat{w}(k) \rightarrow Tx(k) \text{ as } k \rightarrow \infty$$

The estimate $\hat{w}(k)$ of w(k) is given by:

$$\hat{w}(k) = z(k) - Ly(k) \tag{1-22}$$

and the error in an estimate:

$$e(k) = \hat{w}(k) - w(k) \tag{1-23}$$

satisfies the homogeneous equation:

$$e(k+1) = Fe(k) \tag{1-24}$$

1.4 Criteria used in Observer Design

The matrix T in the observer design is chosen to satisfy some design criterion. While the usual procedure is to assign a predetermined set of eigenvalues to the matrix F, we shall in this thesis examine the choice of L to meet the following two criteria:

i) Minimum cost, and

ii) Minimum time.

The first criterion refers to the minimization of a quadratic function of the state estimation error e(k) of the following form:

$$J = \sum_{k=0}^{\infty} e'(k)Qe(k), \qquad Q = Q' > 0 \qquad (1\text{-}25)$$

The quadratic nature of this cost functional penalizes large errors more heavily than small ones and thereby helps to reduce any large overshoots in the estimation errors.

The second criterion concerns the observer which is able to yield the exact state vector of the observed system in the minimum number of steps. This criterion, in general, unique only to the discrete system would be extremely useful when designing a dead-beat controller for the system with some inaccessible state variables.

1.5 Scope of This Thesis

The work reported in the following chapters will mainly focus on the optimal observers in the sense of the minimum time and the minimum cost for the finite dimensional linear time-invariant discrete system.

In chapter two, the design methods developed by Nagata [8] and Porter [10] for the minimum-time observer are reviewed and the similarity between the two methods is discussed.

Chapter three deals with the design method for a minimum-cost observer developed by Gourishankar and Kudva [14].

In chapter four, a simple procedure for designing a minimum-cost, minimum-time observer for the linear discrete system is presented. To the best knowledge of the author this is a new procedure.

In chapter five, the results of simulating both the minimum-cost and the minimum-time observers on a digital computer are discussed.

Chapter six summarizes the results studied in this thesis and states conclusions. A few suggestions for further research are also made.

CHAPTER 2

MINIMUM-TIME OBSERVER

2.1 Introduction

In some applications, we need an observer which is able to reconstruct an errorless estimate of the plant as fast as possible. This is the situation, for example, in the design of the dead-beat controllers with some unknown state variables.

The minimum-time observation problem is similar in concept to the minimum-time control problem, which Kalman et al [5] first considered for the single-input system. The analogous method for the observer was discussed by Ichikawa [6] for a particular multi-output system. Nagata et al [7],[8] generalized this method further and introduced the reduced order observer for the multi-input, multi-output system. A more simplified method using the Luenberger canonical form has been proposed by Porter and Bradshaw [10].

The homogeneous error equation (1-24) for the observer is totally determined by the observer matrix of which eigenvalues can be placed arbitrarily, as long as the observed system is completely observable. If the observer matrix, in particular, has all zero eigenvalues (i.e. nilpotent matrix. Nilpotency is defined in the next section.) the error vanishes in at most

γ steps where γ is the nilpotency index of the observer matrix. This implies that the finite-time observer yields an exact estimate of the plant in at most γ steps. Furthermore, if the nilpotency index γ has the minimum value, the corresponding observer can reconstruct the errorless estimate in minimum time.

2.2 Nilpotent Matrix

Nilpotent matrices play an important role in the minimum-time observer design for the discrete system. In this section, we shall consider the basic concepts relating to and properties of nilpotent matrices. A more complete treatment of this topic can be found elsewhere [11],[12].

Definition: An nxn matrix N is said to be a nilpotent matrix with the index γ if

$$N^{\gamma} = 0 \qquad \text{and} \qquad N^{\gamma-1} \neq 0 \tag{2-1}$$

Theorem 1

If there exists an nx1 vector u such that

$$N^{\gamma-1} u \neq 0 \tag{2-2}$$

then the vectors $u, Nu, N^2u, \ldots, N^{\gamma-1}u$ are linearly independent and γ is at most n (i.e. $\gamma \leq n$).

Proof: Assume that

$$\alpha_1 u + \alpha_2 N u + \alpha_3 N^2 u + \ldots + \alpha_\gamma N^{\gamma-1} u = 0 \tag{2-3}$$

The successive multiplication by N thus gives:

$$\begin{cases} \alpha_1 N u + \alpha_2 N^2 u + \alpha_3 N^3 u + \ldots + \alpha_{\gamma-1} N^{\gamma-1} u = 0 \\ \alpha_1 N^2 u + \alpha_2 N^3 u + \ldots . + \alpha_{\gamma-2} N^{\gamma-1} u = 0 \\ \vdots \\ \alpha_1 N^{\gamma-1} u = 0 \end{cases} \tag{2-4}$$

$N^{\gamma-1} \neq 0$ implies that

$$\alpha_1 = \alpha_2 = \ldots .. = \alpha_\gamma = 0 \tag{2-5}$$

so that the vectors are linearly independent and since more than n vectors in the n-dimensional Euclidian space are always linearly dependent, γ must be equal to or less than n.

Theorem 2

The Jordan canonical form for a nilpotent matrix obtained by a similarity transformation is of the form:

$$T^{-1} N T = \mathrm{diag}\ (N_1, N_2, \ldots ., N_s) \tag{2-6}$$

where the ixi matrices N_i $(i=1,2,\ldots s)$ have the form:

$$N_i = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \tag{2-7}$$

$$\text{Max } i = \gamma; \quad \sum_{i=1}^{s} i = n \tag{2-8}$$

and T is the matrix of the transformation. γ is the nilpotency index of N.

Proof: See Appendix B.

It is clear that a nonzero nilpotent matrix must be singular and the dimension of the largest Jordan block is equal to the nilpotency index.

In the following sections, we shall consider two procedures [8],[10] which make use of the properties of the nilpotent matrices discussed above for the design of minimum-time observers.

2.3 Nagata's Method [8]

The complete observability of the linear discrete system:

$$x(k+1) = Ax(k) + Bu(k); \quad y(k) = Cx(k) \tag{1-1}$$

implies that the rank of the matrix:

$$[C' | A'C' | \ldots\ldots | A'^{n-1}C'] \tag{2-9}$$

is n. This means that there exists a set of linearly independent column vectors[9]:

$$\begin{array}{ll}
c_1', A'c_1', \ldots\ldots, A'^{\sigma_1 - 1}c_1' & :\sigma_1 \text{ vectors} \\
c_2', A'c_2', \ldots\ldots, A'^{\sigma_1 - 1}c_2' & :\sigma_2 \text{ vectors} \\
\vdots \qquad\qquad \vdots & \quad \vdots \qquad \vdots \\
c_m', A'c_m', \ldots\ldots, A'^{\sigma_1 - 1}c_m' & :\sigma_m \text{ vectors}
\end{array} \tag{2-10}$$

where c_i $(i = 1,2,\ldots,m)$ are the i^{th} row vector of C and σ_i are the Kronecker invariants of the pair (A',C'),

and
$$\sum_{i=1}^{m} \sigma_i = n \tag{2-11}$$

The matrix S defined as:

$$S = \begin{array}{l} \left[\begin{array}{l} c_1 \\ c_1 A \\ \vdots \\ c_1 A^{\sigma_1 - 1} \\ c_2 \\ \vdots \\ c_2 A^{\sigma_2 - 1} \\ \vdots \\ c_m A^{\sigma_m - 1} \end{array} \right] \end{array} \begin{array}{l} \\ \\ \\ \leftarrow \mu_1\text{th row} \\ \\ \\ \leftarrow \mu_2\text{th row} \\ \\ \leftarrow \mu_m\text{th row} \end{array} \tag{2-12}$$

becomes nonsingular (i.e. invertible) where

$$\mu_j = \sum_{i=1}^{j} \sigma_i \quad (j = 1,2,\ldots,m) \text{ and } \mu_m = n \tag{2-13}$$

It is convenient to express the matrix S^{-1} in the following manner.

$$S^{-1} = P = [p_{11}|p_{12}|\ldots|p_{1\sigma_1}|p_{21}|\ldots|p_{2\sigma_2}|\ldots|p_{m\sigma_m}] \tag{2-14}$$

where p_{ij} $(i = 1,2,\ldots,m,\ j = 1,2,\ldots,\sigma_i)$ denote the column vectors of the matrix S^{-1}.

Next, define the matrix T as:

$$T = [p_{1\sigma_1}|Ap_{1\sigma_1}|\ldots|\underset{\substack{\uparrow \\ \text{column } \mu_1}}{A^{\sigma_1-1}p_{1\sigma_1}}|p_{2\sigma_2}|\ldots|\underset{\substack{\uparrow \\ \text{column } \mu_2}}{A^{\sigma_2-1}p_{2\sigma_2}}|\ldots|\underset{\substack{\uparrow \\ \text{column } \mu_m}}{A^{\sigma_m-1}p_{m\sigma_m}}] \tag{2-15}$$

It is easily seen that the matrix T is also nonsingular [9]. Then rearrange the column vectors of T to obtain the following matrix W:

$$W = [\underbrace{W_1}_{m_1}|\underbrace{W_2}_{m_2}|\ldots|\underbrace{W_\sigma}_{m_\sigma}] \tag{2-16}$$

where $$W_I = [w_{i1}|w_{i2}|\dots|w_{im_i}] \tag{2-17}$$

$$w_{ij} = A^{\sigma_j - i}\, p_{j\sigma_j} \quad (j = 1,2,\dots,m) \tag{2-18}$$

and $$\sigma = \underset{i}{\text{Max}}\ \sigma_i \quad (i = 1,2,\dots,m)$$

$$\stackrel{\Delta}{=} \text{the observability index [3]} \tag{2-19}$$

If $i \geqslant \sigma_j + 1$, then the particular column vectors w_{ij} do not appear in the matrix W_i. It is obvious by the definition of W that W is nonsingular and

$$m = m_1 \geqslant m_2 \geqslant \dots\dots \geqslant m_\sigma \tag{2-20}$$

$$\sum_{i=1}^{\sigma} m_i = n \tag{2-21}$$

<u>Theorem 3</u>

We define a matrix Ω as

$$\Omega \stackrel{\Delta}{=} A\,(I_n - \phi C) \tag{2-22}$$

where $$\phi = W_1 (CW_1)^{-1} \tag{2-23}$$

then the matrix Ω is a nilpotent matrix with the index σ.

Proof: Clearly T is a Luenberger transformation matrix. Thus $T^{-1}AT$ and CT have the form:

$$T^{-1}AT = \left[\begin{array}{c:c:c:c}
\begin{matrix}0 & \cdots & 0 & \alpha_{11}\\ 1 & \cdots & 0 & \alpha_{21}\\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \alpha_{\mu_1 1}\end{matrix} & \begin{matrix} & \alpha_{12}\\ 0 & \alpha_{22} \\ & \vdots \\ & \alpha_{\mu_1 2}\end{matrix} & \cdots & \begin{matrix} & \alpha_{1m}\\ 0 & \alpha_{2m}\\ & \vdots \\ & \alpha_{\mu_1 m}\end{matrix} \\ \hdashline
\begin{matrix} & \vdots \\ 0 & \vdots \\ & \alpha_{\mu_2 1}\end{matrix} & \begin{matrix}0 & \cdots & 0 & \vdots\\ 1 & & \vdots & \vdots\\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \alpha_{\mu_2 2}\end{matrix} & \cdots & \begin{matrix} & \vdots \\ 0 & \vdots \\ & \alpha_{\mu_2 m}\end{matrix} \\ \hdashline
\vdots & \vdots & \ddots & \vdots \\ \hdashline
\begin{matrix} & \vdots \\ 0 & \vdots \\ & \alpha_{\mu_m 1}\end{matrix} & \begin{matrix} & \vdots \\ 0 & \vdots \\ & \alpha_{\mu_m 2}\end{matrix} & \cdots & \begin{matrix}0 & \cdots & 0 & \vdots\\ 1 & & \vdots & \vdots\\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \alpha_{\mu_m m}\end{matrix}
\end{array}\right] \qquad \text{(2-24a)}$$

↑ column μ_1 ↓ ↑ column μ_2 ↓ ↑ column μ_m ↓

$$CT = \left[\begin{array}{c:c:c:c}
\begin{matrix} & 1 \\ & \beta_{21} \\ 0 & \vdots \\ & \beta_{m1}\end{matrix} & \begin{matrix} & 0 \\ & 1 \\ 0 & \vdots \\ & \beta_{m2}\end{matrix} & & \begin{matrix} & 0 \\ & 0 \\ 0 & \vdots \\ & 1\end{matrix}
\end{array}\right] \qquad \text{(2-24b)}$$

where α_{ij} and $\beta_{j\ell}$ $(i = 1,2,\ldots,m,\ j = 1,2,\ldots,m,\ \ell = 1,2,\ldots,m,\ j \neq \ell)$ denote the possible nonzero elements. Since

W is obtained by simply rearranging the column vectors of the matrix T,

$$T^{-1}W_1 = [e_{\mu_1}|e_{\mu_2}|\ldots|e_{\mu_m}] \tag{2-25}$$

where e_{μ_i} are the μ_ith column vectors of the nxn unit matrix.

$$e_{\mu_i} = \begin{array}{cl} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow \ \mu_i\text{th row } (i = 1,2,\ldots,m) \\ \\ \\ \\ \end{array} \end{array} \tag{2-26}$$

Hence

$$T^{-1}W_1(CW_1)^{-1}CT = [0|..|0|\underset{\substack{\uparrow \\ \text{column } \mu_1}}{e_{\mu_i}}|0|..|0|\underset{\substack{\uparrow \\ \text{column } \mu_2}}{e_{\mu_2}}|0|..|0|\underset{\substack{\uparrow \\ \text{column } \mu_m}}{e_{\mu_m}}] \tag{2-27}$$

Thus $$T^{-1}\ (I_n-\phi C)T = \text{diag}\ (R_1,R_2,\ldots,R_m) \tag{2-28}$$

where the $\sigma_i x \sigma_i$ matrices R_i $(i = 1,2,\ldots,m)$ are of the form:

$$R_i = \left[\begin{array}{c|c} I_{\sigma_i-1} & \\ \hline 0 & 0 \end{array}\right] \tag{2-29}$$

and

$$T^{-1}\Omega T = T^{-1}A(I_n-\phi C)T$$
$$= \text{diag}\ (D_1,D_2,\ldots,D_m) \qquad (2\text{-}30)$$

where the $\sigma_i x\sigma_i$ matrices D_i $(i = 1,2,\ldots,m)$

$$D_i = \left[\begin{array}{c:c} 0 & \\ \hdashline I_{\sigma_i-1} & 0 \end{array}\right] \qquad (2\text{-}31)$$

It is clear that since the matrix $T^{-1}\Omega T$ is a transpose of the nilpotent Jordan canonical form with the index σ, Ω is a nilpotent matrix with the index σ.

Furthermore, the matrix $(I_n-\phi C)A(I_n-\phi C)$ is a nilpotent matrix with the index σ-1 since the multiplication of eqn. (2-28) by eqn. (2-30) gives:

$$T^{-1}(I_n-\phi C)\Omega T = \text{diag}\ (N_1,N_2,\ldots,N_m) \qquad (2\text{-}32)$$

where the $\sigma_i x\sigma_i$ matrices N_i $(i = 1,2,\ldots,m)$

$$N_i = \left[\begin{array}{c:c} 0 & \\ \hdashline I_{\sigma_i-2} & 0 \\ \hdashline 0 & \end{array}\right] \qquad (2\text{-}33)$$

The matrix $(I_n-\phi C)$ with rank n-m is a projection of V onto V_2 along V_1 where V_1 and V_2 are the subspaces of the n dimensional Euclidian space V, that is

$$V = V_1 \oplus V_2 \tag{2-34}$$

and $$C \varepsilon V_1 \tag{2-35}$$

Thus if the rank of the matrix $[C'|J']$ is n so is the rank of the matrix $[C'|T']$ where

$$T = J(I_n-\phi C) \tag{2-36}$$

Theorem 4

The matrix E satisfying

$$ET = I_n-\phi C \tag{2-37}$$

leads to a reduced order minimum-time observer which takes the form:

$$z(k+1) = Fz(k) + Gy(k) + Hu(k) \tag{2-38}$$

where $$F = TAE \tag{2-39a}$$

$$G = TA\phi \tag{2-39b}$$

$$H = TB \tag{2-39c}$$

proof: The solution to the homogeneous estimation error equation:

$$e(k+1) = Fe(k) \tag{2-40}$$

is given:

$$e(k) = F^k e(0) \tag{2-41}$$

Since the matrix T is of full rank, there always exists an nxl vector $\nu(0)$ such that

$$e(0) = T\nu(0) \tag{2-42}$$

Thus substitution of eqns. (2-36), (2-37) and (2-42) into eqn. (2-41) gives:

$$\begin{aligned} e(k) &= T(AET)^k \nu(0) \\ &= J(I_n-\phi C)[A(I_n-\phi C)]^k \nu(0) \end{aligned} \tag{2-43}$$

Since the projection matrix $I_n-\phi C$ is idempotent (i.e. $(I_n-\phi C)(I_n-\phi C) = I_n-\phi C)$, we can write eqn. (2-43) as:

$$e(k) = J[(I_n-\phi C)A(I_n-\phi C)]^k \nu(0) \tag{2-44}$$

Thus the error in the estimate converges to zero in at

most σ-1 steps since $(I_n-\phi C)A(I_n-\phi C)$ is a nilpotent matrix with the nilpotency index σ-1. In other words, the observer governed by eqn. (2-38) produces an exact state estimate of the plant in at most σ-1 steps.

In particular, if J is chosen such that

$$\left[\frac{C}{J}\right] = [\ \phi | E\ ]^{-1} \tag{2-45}$$

where

$$E = [p_{1\sigma_1} | Ap_{1\sigma_1} | \ldots | A^{\sigma_1 -2} p_{1\sigma_1} | p_{2\sigma_2} | \ldots | A^{\sigma_2 -2} p_{2\sigma_2} | \ldots$$

$$\ldots | A^{\sigma_m -2} p_{m\sigma_m}\ ] \tag{2-46}$$

Then $$T = J(I_n-\phi C) = J \tag{2-47}$$

$$ET = I_n - \phi C \tag{2-48}$$

The observer matrix thus has a simple structure as;

$$F = TAE = \mathrm{diag}\ (F_1, F_2, \ldots, F_m) \tag{2-49}$$

where

$$F_i = \left[\begin{array}{c|c} 0 & \\ \hline I_{\sigma_i -1} & 0 \end{array}\right] \quad (i = 1,2,\ldots,m) \tag{2-50}$$

and

$$G = TA\phi = [g_{ij}] \quad (i = 1,2,\ldots,\mu_1-1,\mu_1+1,\ldots\mu_m-1, \quad j = 1,2,\ldots,m) \tag{2-51}$$

where $$g_{ij} = \sum_{\ell=1}^{m} \alpha_{i\ell}\ \gamma_{ij} \tag{2-52}$$

and $$\begin{bmatrix} \gamma_{11} & --- & \gamma_{1m} \\ \gamma_{12} & --- & \gamma_{2m} \\ | & & | \\ \gamma_{m1} & --- & \gamma_{mm} \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ \beta_{21} & \ddots & \\ | & & \\ \beta_{m1} & ---- & 1 \end{bmatrix}^{-1} \tag{2-53}$$

2.4 Porter's Method [10]

In general, a change of coordinates of the system to a canonical form provides a simple structure of the system still possessing all the essential characteristics which can be easily recognized. The observation problem can be facilitated by the Luenberger canonical form, which reduces the m-output system into m separate single-output systems. Thus with the Luenberger canonical matrices we are able to treat the minimum-time observation problem more systematically.

The Porter's method which is described in this section uses the Brunovsky [13] canonical form for designing a minimum-time observer. This canonical form is identical to the Luenberger canonical form.

Let the triple $(\bar{C}, \bar{A}, \bar{B})$ be in the Luenberger canonical form, that is

$$\bar{A} = T^{-1}AT \tag{2-24a}$$

$$\bar{B} = T^{-1}B \tag{2-54}$$

$$\bar{C} = CT \tag{2-24b}$$

and $$\bar{x}(k) = T^{-1}x(k) \tag{2-55}$$

Then, with the simple structure of the matrices $\bar{C}$ and $\bar{A}$ we can describe the system in the following manner.

$$\left\{\begin{array}{lllll}
\bar{x}_1(k+1) & = \sum_{i=1}^{m} \alpha_{1i}\bar{x}_{\mu_i}(k) & + \sum_{i=1}^{p} b_{1i}u_i & & \\
\bar{x}_2(k+1) & = \sum_{i=1}^{m} \alpha_{2i}\bar{x}_{\mu_i}(k) & + \sum_{i=1}^{p} b_{2i}u_i & + & \bar{x}_1(k) \\
\vdots & \quad\vdots & \quad\vdots & & \vdots \\
\bar{x}_{\mu_1}(k+1) & = \sum_{i=1}^{m} \alpha_{\mu_1 i}\bar{x}_{\mu_i}(k) & + \sum_{i=1}^{p} b_{\mu_1 i}u_i & + & \bar{x}_{\mu_1 -1}(k) \\
\vdots & \quad\vdots & \quad\vdots & & \vdots \\
\bar{x}_{\mu_m}(k+1) & = \sum_{i=1}^{m} \alpha_{\mu_m i}\bar{x}_{\mu_i}(k) & + \sum_{i=1}^{p} b_{\mu_m i}u_i & + & \bar{x}_{\mu_m -1}(k)
\end{array}\right. \tag{2-56a}$$

$$\begin{cases} y_1(k) = \bar{x}_{\mu_1}(k) \\ y_2(k) = \beta_{21}\bar{x}_{\mu_1}(k) + \bar{x}_{\mu_2}(k) \\ \vdots \\ y_m(k) = \sum_{i=1}^{m-1} \beta_{mi}\bar{x}_{\mu_i}(k) + \bar{x}_{\mu_m}(k) \end{cases} \tag{2-56b}$$

where

$$\bar{x}(k) = \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \\ \vdots \\ \bar{x}_{\mu_m}(k) \end{bmatrix}, \quad y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_m(k) \end{bmatrix}, \quad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_p(k) \end{bmatrix} \tag{2-57}$$

and

$$\bar{B} = \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2p} \\ \vdots & & \vdots \\ b_{\mu_m 1} & \cdots & b_{\mu_m p} \end{bmatrix} \tag{2-58}$$

Since the state variables $\bar{x}_i$ $(i = \mu_1, \mu_2, \ldots, \mu_m)$ are determined through the measurement of the outputs $y(k)$, another n-m dimensional dynamic system (observer) described by:

$$
\begin{cases}
z_1(k+1) = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_{1i}\gamma_{ij}y_j(k) + \sum_{i=1}^{p} b_{1i}u_i(k) \\
z_2(k+1) = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_{2i}\gamma_{ij}y_j(k) + \sum_{i=1}^{p} b_{2i}u_i(k) + z_1(k) \\
\vdots \\
z_{\mu_1-1}(k+1) = \sum_{i=1}^{m} \sum_{j-1}^{m} \alpha_{\mu_1-1i}\gamma_{ij}y_j(k) + \sum_{i=1}^{p} b_{\mu_1-1i}u_i(k) + z_{\mu_1-2}(k) \\
z_{\mu_1+1}(k+1) = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_{\mu_1+1i}\gamma_{ij}y_j(k) + \sum_{i=1}^{p} b_{\mu_1+1i}u_i(k) \\
\vdots \\
z_{\mu_m-1}(k+1) = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_{\mu_m-1i}\gamma_{ij}y_j(k) + \sum_{i=1}^{p} b_{\mu_m-1i}u_i(k) + z_{\mu_m-2}(k)
\end{cases}
\tag{2-59}
$$

where γ_{ij} $(i = 1,2,\ldots,m,\ j = 1,2,\ldots,m)$ are the elements of the mxm matrix:

$$
\begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \gamma_{21} & \cdots & \gamma_{2m} \\ \vdots & & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mm} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \beta_{21} & 1 & & 0 \\ \vdots & & \ddots & \\ \beta_{m1} & \cdots & \cdots & 1 \end{bmatrix}^{-1} \tag{2-60}
$$

thus produces an exact estimate of the state variables x_{μ_i+j} $(i = 0,1,\ldots,m-1,\ j = 1,2,\ldots,\sigma_{i+1}-1,\ \mu_0 = 0)$ in at most j steps. Hence the observer yields the inaccessible state vector in at most $\sigma-1$ steps.

The representation of eqn. (2-59) in the matrix form is:

$$z(k+1) = Fz(k) + Gy(k) + Hu(k)$$

$$= \begin{bmatrix} F_1 & & & 0 \\ & F_2 & & \\ & & \ddots & \\ 0 & & & F_m \end{bmatrix} z(k) + \begin{bmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{\mu_1-11} & \cdots & g_{\mu_1-1m} \\ g_{\mu_1+11} & \cdots & g_{\mu_1+1m} \\ \vdots & & \vdots \\ g_{\mu_m-11} & \cdots & g_{\mu_m-1m} \end{bmatrix} y(k)$$

$$+ \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{\mu_1-11} & \cdots & b_{\mu_1-1p} \\ b_{\mu_1+11} & \cdots & b_{\mu_1+1p} \\ \vdots & & \vdots \\ b_{\mu_m+11} & \cdots & b_{\mu_m+1p} \end{bmatrix} u(k) \qquad (2\text{-}61)$$

where the $\sigma_i-1 \times \sigma_i-1$ matrices F_i $(i = 1,2,\ldots m)$ have the form:

$$F_i = \left[\begin{array}{c|c} 0 & \\ \hline I_{\sigma_i-2} & 0 \end{array}\right] \qquad (2\text{-}62)$$

and

$$g_{ij} = \sum_{\ell=1}^{m} \alpha_{i\ell}\gamma_{\ell j} \quad (i = 1,2,\ldots\mu_1-1,\mu_1+1,\ldots,\mu_m-1,$$
$$j = 1,2,\ldots,m) \qquad (2\text{-}63)$$

Comparing eqns. (2-62), (2-63) with eqns. (2-49), (2-51) it can be easily seen that the observer governed by eqn. (2-59) is a particular case of eqn. (2-38).

Computer simulations to illustrate the performance of Nagata's and Porter's minimum-time observers are presented in Chapter 5.

CHAPTER 3

THE MINIMUM COST OBSERVER

3.1 Problem Formulation

Ideally, we would like to design an observer which estimates the plant state vector without any error. However, this cannot, in general, be achieved since the initial condition on the unmeasurable plant state variables is not known. Thus it is desirable, as a compromise, to design an observer which estimates the unmeasurable plant state as closely as possible at every step. In practice, in order to realize this, we often formulate a mathematical performance index which reasonably describes the desired goal and then minimize this particular performance measure. A useful performance index in the case of discrete observers is the following quadratic function of the state estimation errors (see eqn. 1-25):

$$J = \sum_{k=0}^{\infty} e'(k)Qe(k) \qquad Q = Q' > 0. \tag{1-25}$$

where e(k) is as defined in eqn. (1-23). A minimization of this criterion, suppresses any large deviations from the state of the plant and thereby forces the observer to produce an estimate as close as possible to the unmeasurable state of the plant.

In the following sections, a design procedure for a minimum-cost observer (minimum in an average sense), developed by Gourishankar and Kudva [14] is presented.

3.2 Optimality in an Average Cost

Again the system under consideration is assumed to be

$$x(k+1) = Ax(k) + Bu(k); \quad y(k) = Cx(k) \tag{1-13}$$

and

$$C = [I_m \mid 0] \tag{1-14}$$

An observer [4] reconstructing the unavailable part of the state vector, w, can be designed as follows.

$$z(k+1) = Fz(k) + Gy(k) + Hu(k) \tag{1-20}$$

where

$$F = A_{22} + LA_{12} \tag{1-21a}$$

$$G = FL + A_{21} + LA_{11} \tag{1-21b}$$

$$H = B_2 + LB_1 \tag{1-21c}$$

$$A = \begin{array}{c} \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{array}{l} \}\, m \\ \}\, n-m \end{array} \\ \begin{array}{cc} \underbrace{\phantom{A_{21}}}_{m} & \underbrace{\phantom{A_{22}}}_{n-m} \end{array} \end{array} \qquad B = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] \begin{array}{l} \}\, m \\ \}\, n-m \end{array} \tag{1-16}$$

and the elements of the matrix L are the free design parameters subject to the stability of the observer.

Then the estimate $\hat{w}(k)$ of $w(k)$ is given by

$$\hat{w}(k) = z(k) - Ly(k) \tag{1-22}$$

Thus the estimation error:

$$e(k) = \hat{w}(k) - w(k) \tag{1-23}$$

satifies the homogeneous difference equation:

$$e(k+1) = Fe(k) \tag{1-24}$$

The quadratic function (1-25) becomes

$$J = \sum_{k=0}^{\infty} [F^k e(0)]' Q F^k e(0) \tag{3-1}$$

If we let

$$S = \sum_{k=0}^{\infty} (F')^k Q F^k \tag{3-2}$$

and assume F to be a stable matrix, it immediately follows that the matrix S is the solution to:

$$F'SF - S + Q = 0 \tag{3-3}$$

Therefore, the quadratic function is expressed by

$$\begin{aligned} J &= e'(0)Se(0) \\ &= [\hat{w}(0) - w(0)]'S[\hat{w}(0) - w(0)] \qquad (3\text{-}4) \\ &= [z(0) - Ly(0) - w(0)]'S[z(0) - Ly(0) - w(0)] \end{aligned}$$

where S is the positive definite matrix satisfying eqn. (3-3).

Further, if the initial condition on the observer state is chosen to be

$$z(0) = Ly(0) \tag{3-5}$$

so that

$$\hat{w}(0) = 0 \tag{3-6}$$

Eqn. (3-4) simply becomes

$$J = w'(0)Sw(0) \tag{3-7}$$

Since w(0) is unknown, the explicit dependence of the performance index on w(0) should be avoided in determining

an optimal observer matrix F.

A simple way [1] to eliminate this dependence is to consider the average of the performance obtained for a linearly independent set of values for w(0). Mathematically this is equivalent to assuming w(0) to be a random vector uniformly distributed on the surface of the n-m dimensional unit sphere. Then the expected value of $\hat{J}$ of the performance index is given by

$$(n-m)\ \hat{J} = \sum_{i=1}^{n-m} e_i' S e_i = tr(S) \tag{3-8}$$

where the vectors e_i $(i = 1,2,...,n-m)$ are the i th columns of the $(n-m) \times (n-m)$ unit matrix and S is as defined earlier.

This average performance index, which is independent of the initial state, enables us to obtain an optimal observer in an average sense.

3.3 Optimal Observer Design

The optimal design problem is thus converted to one involving the selection of the matrix L which minimizes tr(S), subject to the constraint eqn. (3-3). However, in general, this problem leads to a solution which involves the inversion of singular matrices. Thus in order to avoid this difficulty, the performance measure tr(S) is augmented with the term $\alpha tr(L'SL)$, $\alpha>0$ and the

following modified criterion is minimized with respect to the matrices L and S:

$$\mathrm{tr}(S) + \alpha\ \mathrm{tr}(L'SL) \tag{3-9}$$

subject to the constraint equation:

$$(A_{22} + LA_{12})'S(A_{22} + LA_{12}) - S + Q = 0 \tag{3-10}$$

This problem can be formulated as an unconstrained minimization of the augmented criterion:

$$\begin{aligned} J_1 = \mathrm{tr}[S + \Omega\{(A_{22} + LA_{12})'S(A_{22} + LA_{12}) \\ - S + Q\}] + \alpha\ \mathrm{tr}[L'SL] \end{aligned} \tag{3-11}$$

where Ω is a symmetric positive definite Lagrange multiplier matrix.

It has been shown [14] that the solution to this problem, which also ensures the stability of the observer, possesses limiting values as $\alpha \to 0^+$ and the limiting value of L is:

$$L_o^* = - A_{22}\Omega_o^* A_{12}' \ (A_{12}\Omega_o^* A_{12}')^+ \tag{3-12}$$

where Ω_o^* is the positive definite solution to

$$\Omega_o^* = A_{22}\Omega_o^* A_{22}' - A_{22}\Omega_o^* A_{12}' \, (A_{12}\Omega_o^* A_{12}')^+$$

$$A_{12}\Omega_o^* A_{22}' + I_{n-m} \tag{3-13}$$

The determination of L_o^* and Ω_o^* from eqns. (3-12) and (3-13) can be broken down into two cases:

a) the matrix A_{12} is of full rank, and

b) A_{12} is not of full rank

Case a) A_{12} is of full rank

i) if $n > 2m$, then, $(A_{12}\Omega_o^* A_{12}')$ is nonsingular and the matrix Ω_o^* can be directly obtained by solving the Riccati equation:

$$\Omega_o^* = A_{22}\Omega_o^* A_{22}' - A_{22}\Omega_o^* A_{12}' \, (A_{12}\Omega_o A_{12}')^{-1}$$

$$A_{12}\Omega_o^* A_{22}' + I_{n-m} \tag{3-14}$$

and L_o^* is to be

$$L_o^* = - A_{22}\Omega_o^* A_{12}' \, (A_{12}\Omega_o^* A_{12}')^{-1} \tag{3-15}$$

ii) if $n \leqslant 2m$ by using the matrix inversion lemma it is easily seen that:

$$\Omega^* A_{12}' \, (\alpha I_m + A_{12}\Omega^* A_{12}')^{-1} =$$

$$(A_{12}'A_{12} + \alpha\Omega^{*-1})^{-1} A_{12}' \tag{3-16}$$

Hence, taking the limit as $\alpha \to 0^+$ eqns. (3-12) and (3-13) become

$$L_o^* = - A_{22} (A_{12}'A_{12})^{-1} A_{12}' \qquad (3\text{-}17)$$

and $$\Omega_o^* = I_{n-m} \qquad (3\text{-}18)$$

Case b) A_{12} is not of full rank

In this case the solution of eqn. (3-13) for Ω_o^* is, in general, not straightforward except for lower-order systems. The following numerical procedure has been proposed [14]:

i) With an arbitrary small positive value of α solve the Riccati equation:

$$\Omega = A_{22}\Omega A_{22}' - A_{22}\Omega A_{12}' (\alpha I_m + A_{12}\Omega A_{12}')^{-1} A_{12}\Omega A_{22}' + I_{n-m} \qquad (3\text{-}19)$$

for a positive definite matrix Ω,

ii) then evaluate

$$\phi = A_{22}\Omega A_{22}' - A_{22}\Omega A_{12}' (A_{12}\Omega A_{12}')^+ A_{12}\Omega A_{22}' + I_{n-m} \qquad (3\text{-}20)$$

using the value Ω obtained in step (i)

iii) if $\| \Omega - \phi \| < \varepsilon$ where ε is a preselected small positive constant, set $\Omega_o^* = \Omega$ and compute L_o^* from

eqn. (3-12); otherwise decrease the value of α and repeat the above procedure.

Once the matrix L_o^* is obtained, then the design of the optimal observer is completed by computing the optimal observer matrices F^*, G^* and H^* from eqn. (1-21) and setting the initial condition on the observer state $z(0) = L_o^* y(0)$.

Example:

Consider a fifth-order linear discrete system with

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Here,

$$A_{22} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{rank } (A_{12}) = 1$$

and let

$$\Omega_o^* = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{12} & \omega_{22} & \omega_{23} \\ \omega_{13} & \omega_{23} & \omega_{33} \end{bmatrix}$$

then,

$$A_{12}\Omega_o A_{12}' = \begin{bmatrix} \omega_{11} + 2\omega_{13} + \omega_{33}, & \omega_{11} + 2\omega_{13} + \omega_{33} \\ \omega_{11} + 2\omega_{13} + \omega_{33}, & \omega_{11} + 2\omega_{13} + \omega_{33} \end{bmatrix}$$

and

$$(A_{12}\Omega_o A_{12}')^+ = \frac{1}{4(\omega_{11} + 2\omega_{13} + \omega_{33})} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Thus, the solution to eqn. (3-17) for Ω_o^* is given as:

$$\Omega_o^* = \begin{bmatrix} 2.49 & -0.25 & 1.74 \\ -0.25 & 1.62 & -0.87 \\ 1.74 & -0.87 & 3.61 \end{bmatrix}$$

From eqn. (3-16),

$$L_o^* = \begin{bmatrix} -0.44 & -0.44 \\ -0.28 & -0.28 \\ -0.16 & -0.16 \end{bmatrix}$$

so that from eqn. (1-21) the optimal observer matrices are

$$F^* = \begin{bmatrix} 0.12 & 1 & 0.12 \\ -0.56 & 0 & 0.44 \\ 0.67 & 1 & -0.33 \end{bmatrix}$$

$$G^* = \begin{bmatrix} -0.09 & -0.09 \\ -0.45 & 0.55 \\ 0.36 & 0.36 \end{bmatrix}$$

$$H^* = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

It will be shown in the next chapter that the optimal observer design procedure described in the preceding sections can be considerably simplified if the system is represented in a suitable canonical form. In such a case, a direct solution to the matrix Riccati eqn. (3-13) can be obtained whether the matrix A_{12} is of full rank or not. It will also be shown that the resulting observer is not only a minimum-cost observer but also a minimum-time observer.

CHAPTER 4

MINIMUM-TIME MINIMUM-COST OBSERVER

4.1 Canonical Representation for Observer Design

In the previous chapter, a procedure was presented for the design of an observer such that a quadratic function of the state estimation error was minimized. In this chapter it will be shown that this design procedure can be considerably simplified by the choice of a suitable canonical representation for the system. Interestingly enough, the resulting observer is found to also possess the minimum-time state estimation property.

Let the discrete system (1-1) be represented by

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k); \; y(k) = \bar{C}\bar{x}(k) \tag{4-1}$$

in the Luenberger canonical form. This assumption can be made without any loss of generality. It follows that

$$\bar{A} = T^{-1}AT \tag{2-24a}$$

$$\bar{C} = CT \tag{2-24b}$$

$$\bar{B} = T^{-1}B \tag{2-54}$$

and

$$\bar{x}(k) = T^{-1}x(k) \tag{2-55}$$

The observer design is facilitated by introducing a suitable rearrangement of the plant state variables. This is achieved by the following coordinate transformation [15]:

$$\underline{x}(k) = P\bar{x}(k) \tag{4-2}$$

This transformation enables the system in the new coordinates to retain the simplicity of the essential structural feature in the matrices $\bar{A}$ and $\bar{C}$, where

i) the nxn matrix P is given by

$$P = \begin{bmatrix} \bar{C} \\ \hline D \end{bmatrix} \tag{4-3}$$

ii) the (n-m)xn matrix D is given by

$$D = \text{diag}\ (D_1, D_2, \ldots, D_m) \tag{4-4}$$

where the $(\sigma_i - 1) x \sigma_i$ matrices D_i $(i = 1, 2, \ldots, m)$ are given by

$$D_i = \left[I_{\sigma_i - 1} \;\vdots\; 0 \right] \tag{4-5}$$

and σ_i $(i = 1, 2, \ldots, m)$ are the Kronecker invariants of the pair (A', C').

The transformed form of eqn. (4-1) can be written as:

$$\underline{x}(k+1) = \underline{A}\underline{x}(k) + \underline{B}u(k); \quad y(k) = \underline{C}\underline{x}(k) \tag{4-6}$$

where:

i)

$$\underline{A} = P\bar{A}P^{-1} = \begin{array}{c} \left[\begin{array}{c|c} \underline{A}_{11} & \underline{A}_{12} \\ \hline \underbrace{\underline{A}_{21}}_{m} & \underbrace{\underline{A}_{22}}_{n-m} \end{array}\right] \end{array} \begin{array}{l} \}\, m \\ \}\, n-m \end{array} \tag{4-7}$$

$$\underline{A}_{12} = \left[\underbrace{0 \mid \bar{C}_{\mu_1}}_{\sigma_1 - 1} \mid \underbrace{0 \mid \bar{C}_{\mu_2}}_{\sigma_2 - 1} \mid \cdots \mid \underbrace{0 \mid \bar{C}_{\mu_m}}_{\sigma_m - 1} \right] \tag{4-8}$$

$\bar{C}_{\mu_i}$ $(i = 1,2,\ldots,m)$ are the μ_ith column vectors of $\bar{C}$

$$\mu_i = \sum_{j=1}^{i} \sigma_i \tag{2-13}$$

$$\underline{A}_{21} = \begin{bmatrix} g_{11} & g_{12} & --- & g_{1m} \\ \vdots & \vdots & & \vdots \\ g_{\mu_1-11} & g_{\mu_1-12} & --- & g_{\mu_1-1m} \\ g_{\mu_1+11} & g_{\mu_1+12} & --- & g_{\mu_1+1m} \\ \vdots & \vdots & & \vdots \\ g_{\mu_m-11} & g_{\mu_m-12} & --- & g_{\mu_m-1m} \end{bmatrix} \tag{4-9}$$

$$g_{ij} = \sum_{\ell=1}^{m} \alpha_{i\ell}\gamma_{\ell j} \quad (i = 1,2,\ldots,\mu_1-1,\mu_1+2,\ldots,\mu_m-1, \quad j = 1,2,\ldots,m) \tag{2-63}$$

$$\underline{A}_{22} = \text{diag}\ (E_1,E_2,\ldots,E_m) \tag{4-10}$$

the $(\sigma_i-1)x(\sigma_i-1)$ matrices M_i are given by

$$E_i = \left[\begin{array}{c|c} 0 & \\ \hline I_{\sigma_i-2} & 0 \end{array}\right] \quad (i = 1,2,\ldots,m) \tag{4-11}$$

ii) $$\underline{C} = \overline{C}P^{-1} = \left[\begin{array}{c|c} I_m & 0 \end{array}\right] \tag{4-12}$$

iii) $$\underline{B} = P\overline{B} = \begin{array}{c} \left[\begin{array}{c} \underline{B}_1 \\ \hline \underline{B}_2 \end{array}\right] \begin{array}{l} \}\ m \\ \}\ n-m \end{array} \end{array} \tag{4-13}$$

and

iv) the state vector is

$$\underline{x}(k) = \left[\begin{array}{c} y(k) \\ \hline w(k) \end{array}\right] \begin{array}{l} \}\ m \\ \}\ n-m \end{array} \tag{4-14}$$

y(k) being the accessible output vector and w(k), the unmeasurable part of the state vector.

4.2 Optimal Observer

Since the system equation (4-6) is the same in the form as eqn. (1-13), we can follow the design procedure described in the previous chapter to obtain an optimal observer for estimating w(k). This observer is a minimum-cost observer in an average sense as defined in Chapter 3.

$$z(k+1) = F^* z(k) + G^* y(k) + H^* u(k) \tag{4-15}$$

where

$$F^* = \underline{A}_{22} + L_o^* \underline{A}_{12} \tag{4-16a}$$

$$G^* = FL_o^* + \underline{A}_{21} + L_o^* \underline{A}_{11} \tag{4-16b}$$

$$H^* = \underline{B}_2 + L_o^* \underline{B}_1 \tag{4-16c}$$

$$L_o^* = -\underline{A}_{22}\Omega_o^* \underline{A}_{12}' \, (\underline{A}_{12}\Omega_o^* \underline{A}_{12}')^+ \tag{4-17}$$

and Ω_o^* is the positive definite solution to the Riccati equation:

$$\Omega = \underline{A}_{22}\Omega\underline{A}_{22}' - \underline{A}_{22}\Omega\underline{A}_{12}' \, (\underline{A}_{12}\Omega\underline{A}_{12}')^+ \, \underline{A}_{12}\Omega\underline{A}_{22}' + I_{n-m} \tag{4-18}$$

and the initial condition on the observer state is chosen as

$$z(0) = L_o^* y(0) \tag{4-19}$$

The design procedure thus requires the solution to the Riccati eqn. (4-18). We shall consider solving eqn. (4-18) by the following iterative algorithm:

$$\Omega_{i+1} = \underline{A}_{22}\Omega_i\underline{A}_{22}' - \underline{A}_{22}\Omega_i\underline{A}_{12}' (\underline{A}_{12}\Omega_i\underline{A}_{12}')^{+} \underline{A}_{12}\Omega_i\underline{A}_{22}' + I_{n-m} \tag{4-20}$$

with $\Omega_0 = I_{n-m}$.

In view of the simplicity of the structure of the $\underline{A}_{12}$ and $\underline{A}_{22}$, it can be easily shown that

$$\text{i)} \quad \underline{A}_{22}^{\sigma-1} = 0 \tag{4-21}$$

$$\text{ii)} \quad \underline{A}_{22}^{j} (\underline{A}_{22}')^{j} = \text{diag}\,(M_1, M_2, \ldots, M_m) \qquad j = 1,2,\ldots,\sigma-2 \tag{4-22}$$

where the $(\sigma_i-1)\times(\sigma_i-1)$ matrices M_i are

$$M_i = \left[\begin{array}{c|c} 0 & 0 \\ \hline & I_{\sigma_i-j-1} \end{array}\right] \quad i = 1,2,\ldots,m \tag{4-23}$$

and

$$\text{iii)} \quad \underline{A}_{22}\underline{A}_{12}' = 0 \tag{4-24}$$

with the particular structures of $\underline{A}_{22}{}^{j}$ $(\underline{A}_{22}')^{j}$ and $\underline{A}_{12}$, it is also easily seen that

$$\underline{A}_{22}{}^{j}\ (\underline{A}_{22}')^{j}\ \underline{A}_{12}' = \underline{A}_{12}' \quad (j = 1,2,\ldots,\sigma-2) \qquad (4\text{-}25)$$

Thus using eqn. (4-24)

$$\underline{A}_{22}{}^{j}\ (\underline{A}_{22}')^{j-1}\ \underline{A}_{12}' = 0 \qquad (4\text{-}26)$$

By using eqn. (4-25) the sequence of Ω_i can be expressed as follows:

$$\begin{aligned}
\Omega_0 &= I_{n-m} \\
\Omega_1 &= \underline{A}_{22}\underline{A}_{22}' + I_{n-m} \\
\Omega_2 &= (\underline{A}_{22})^2(\underline{A}_{22}')^2 + \underline{A}_{22}\underline{A}_{22}' + I_{n-m} \qquad (4\text{-}27) \\
\\
\Omega_i &= (\underline{A}_{22})^i(\underline{A}_{22}')^i + (\underline{A}_{22})^{i-1}(\underline{A}_{22}')^{i-1} + \ldots \\
&\quad \ldots + A_{22}A_{22}' + I_{n-m}
\end{aligned}$$

Since $\underline{A}_{22}{}^{\sigma-1} = 0$, it is clearly obvious that this algorithm terminates at the $(\sigma-2)$th step to give

$$\Omega_o{}^{*} = \Omega_{\sigma-2} = \sum_{j=1}^{\sigma-2} (\underline{A}_{22})^{j}(\underline{A}_{22}')^{j} + I_{n-m} \qquad (4\text{-}28)$$

The matrix $\Omega_o{}^{*}$ can now be directly computed from eqns. (4-22) and (4-28) and is obtained as:

$$\Omega_o^* = \left[\begin{array}{c|c|c|c} \begin{matrix}1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & \sigma_1-1\end{matrix} & 0 & \cdots & 0 \\ \hline 0 & \begin{matrix}1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & \sigma_2-1\end{matrix} & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & \begin{matrix}1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & \sigma_m-1\end{matrix} \end{array}\right] \qquad (4\text{-}29)$$

<u>4.3 A Comparison with the Minimum-Time</u>

With the value of Ω_o^* in eqn. (4-29), we have

$$\underline{A}_{22}\Omega_o^*\underline{A}_{12}' = 0 \qquad (4\text{-}30)$$

and hence from eqn. (4-17)

$$L_o^* = 0 \qquad (4\text{-}31)$$

Thus the optimal observer, in an average sense, obtained from eqns. (4-15), (4-16), (4-18) and (4-31) is simply expressed as:

$$z(k+1) = \underline{A}_{22}z(k) + \underline{A}_{21}y(k) + \underline{B}_2u(k) \tag{4-32}$$

with $z(0) = 0$

and the estimate $\hat{w}(k)$ of w is given by

$$\hat{w}(k) = z(k)$$

This minimum-cost observer being independent of the matrix Q, reveals several interesting features both in the design procedure and in itself. By introducing a particular canonical representation for the system (1-1), the design procedure is considerably simplified in comparison to that discussed in the previous chapter. The procedure, here, involves merely selecting the appropriate submatrices from the pair $(\underline{A},\underline{B})$. Comparing eqns. (2-62) and (2-63) with eqns. (4-9) and (4-10) the minimum-cost observer obtained in this chapter is identical to the minimum-time observer obtained by Porter's method. Thus the resulting observer produces an errorless estimate of the plant state vector in the minimum-time [10] as well as minimizes, in an average sense, the performance index. We

shall refer to it as a minimum-cost, minimum-time observer.

Simulation results to demonstrate the performance of this observer are presented in Chapter 5.

4.4 Example:

Consider a fifth-order linear discrete system with

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The Luenberger canonical form is then given as:

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

with $\sigma_1 = 4$ and $\sigma_2 = 1$

Thus the transformation matrix P having the form:

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

changes the coordinates such that

$$\underline{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ -2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\underline{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \underline{A}_{21} = \begin{bmatrix} -2 & 3 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Here,

$$\underline{A}_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \underline{A}_{12} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \underline{B}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the solution to eqn. (4-18) for Ω_o^* is given by iterative algorithm (4-20) as follows

$$\Omega_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Omega_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Omega_o^* = \Omega_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

From eqn (4-17)

$$L_o^* = 0$$

so that from eqn (4-16) the optimal observer matrices are

$$F^* = \underline{A}_{22}$$

$$G^* = \underline{A}_{21}$$

and

$$H^* = \underline{B}_2$$

CHAPTER 5

SIMULATION RESULTS

5.1 Introduction

In this chapter, the objective is to present computer simulation results to compare the effectiveness of the three observers discussed in the preceding chapters. This comparison will be made in two ways. The first test will involve the comparison of the estimates generated by each of the three observers of the actual states of a force free dynamic plant in an open-loop configuration. In the second test the observers will be incorporated in a closed loop system along with a feedback controller designed to minimize a quadratic performance criterion, assuming complete state feedback.

In order to obtain meaningful comparisons the same plant is used in all the tests. The plant is of fifth order with 2 outputs. It will be assumed that 3 of 5 states are unmeasurable. A third order observer will be needed to estimate these variables. The plant matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 2 & 1 & -1 \\ 2 & 1 & 4 & 4 & 2 \end{bmatrix}$$

The initial value of the state vector x will be assumed to be

$$x(0) = \begin{bmatrix} -2 \\ 1 \\ -4 \\ -2 \\ 4 \end{bmatrix}$$

5.2 Open-loop Test

Referring to Chapter 3 and 4 the initial conditions on the states of Observers C* and TC are computed

---

* For the sake of convenience, the minimum observer designed by Nagata, et al will be referred to as Observer T. The minimum-time observer of Porter and the minimum-time and minimum-cost observer designed by the author which are essentially the same as Observer TC and the minimum-cost observer designed by Gourishankar and Kudva Observer C.

to be

$$z_C(0) = [23.31,\ 31.46,\ 50.61]'$$

and

$$z_{TC}(0) = [0,\ 0,\ 0]'$$

Since the initial value of the state vector of Observer T is not specified in the design procedure the initial state is arbitrarily chosen to be

$$z_T(0) = [1,\ -4,\ 3]'$$

The corresponding composite open-loop system responses are shown in Fig. 2 in terms of the error between the estimates and the actual values of the state.

It may be useful to point out the characteristic properties in behavior of the system.

1) Interestingly enough, the estimation error of the system for Observer C converges close enough to zero in one step less compared to the other observers although the error does not vanish in finite steps, and is accompanied by large overshoots in all the state variables.

2) Also as expected, Observers T and TC start to yield the errorless state of the plant in σ-1 step which is 2 step.

3) Comparing the magnitude of the overshoots of the

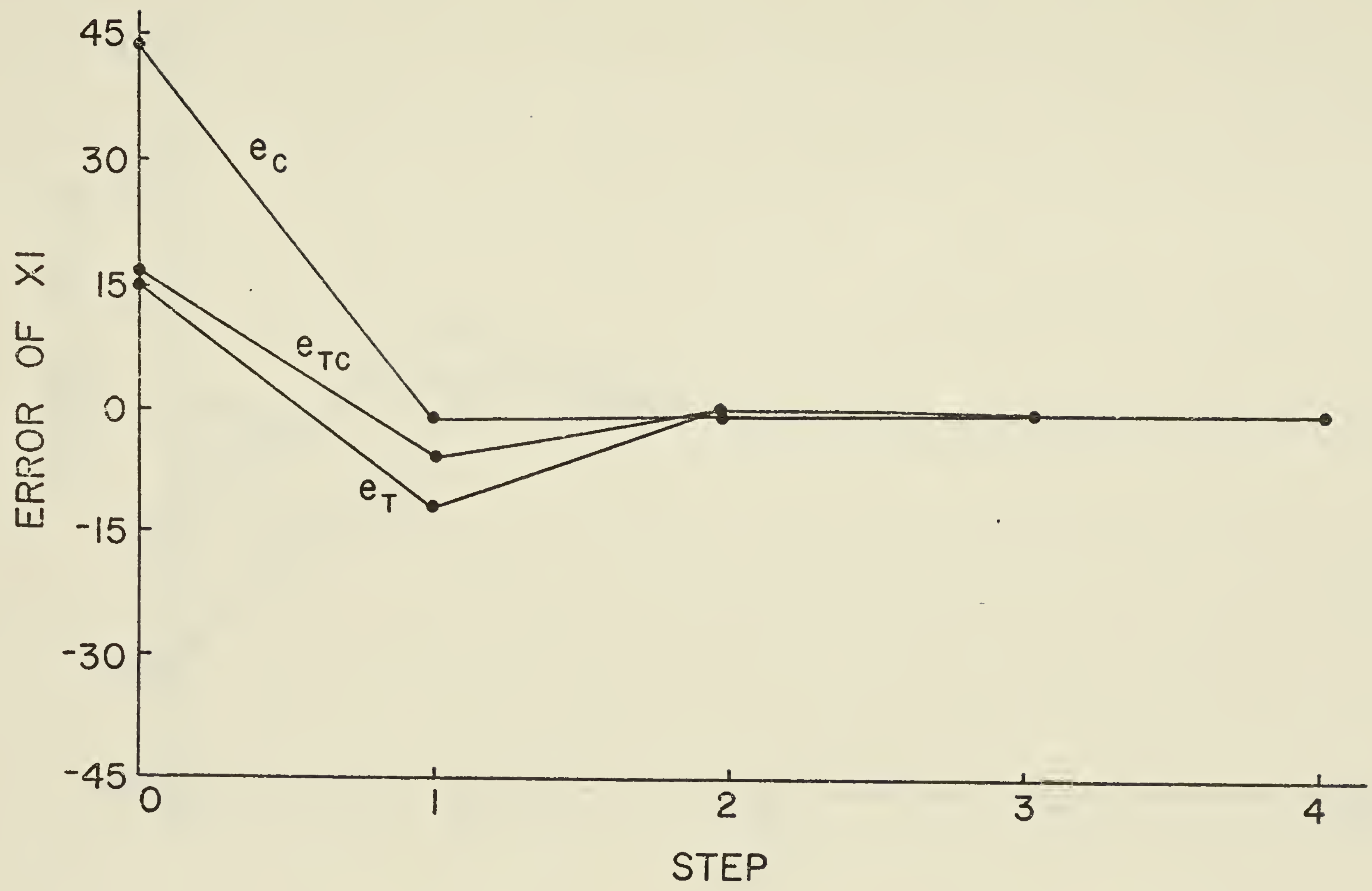

Figure 2(a): Open-Loop State Responses for $\hat{x}_1$

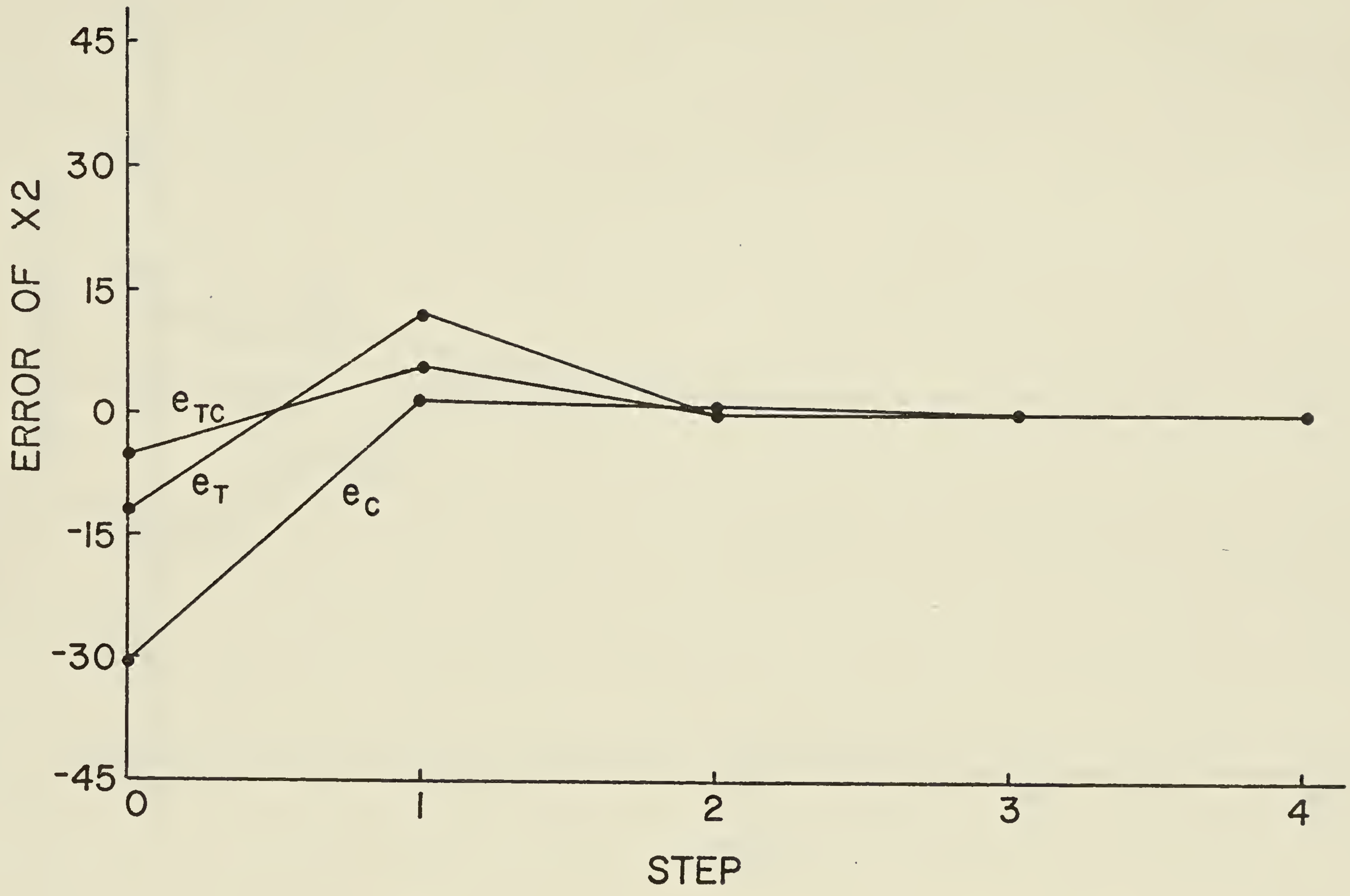

Figure 2(b): Open-Loop State Responses for $\hat{x}_2$

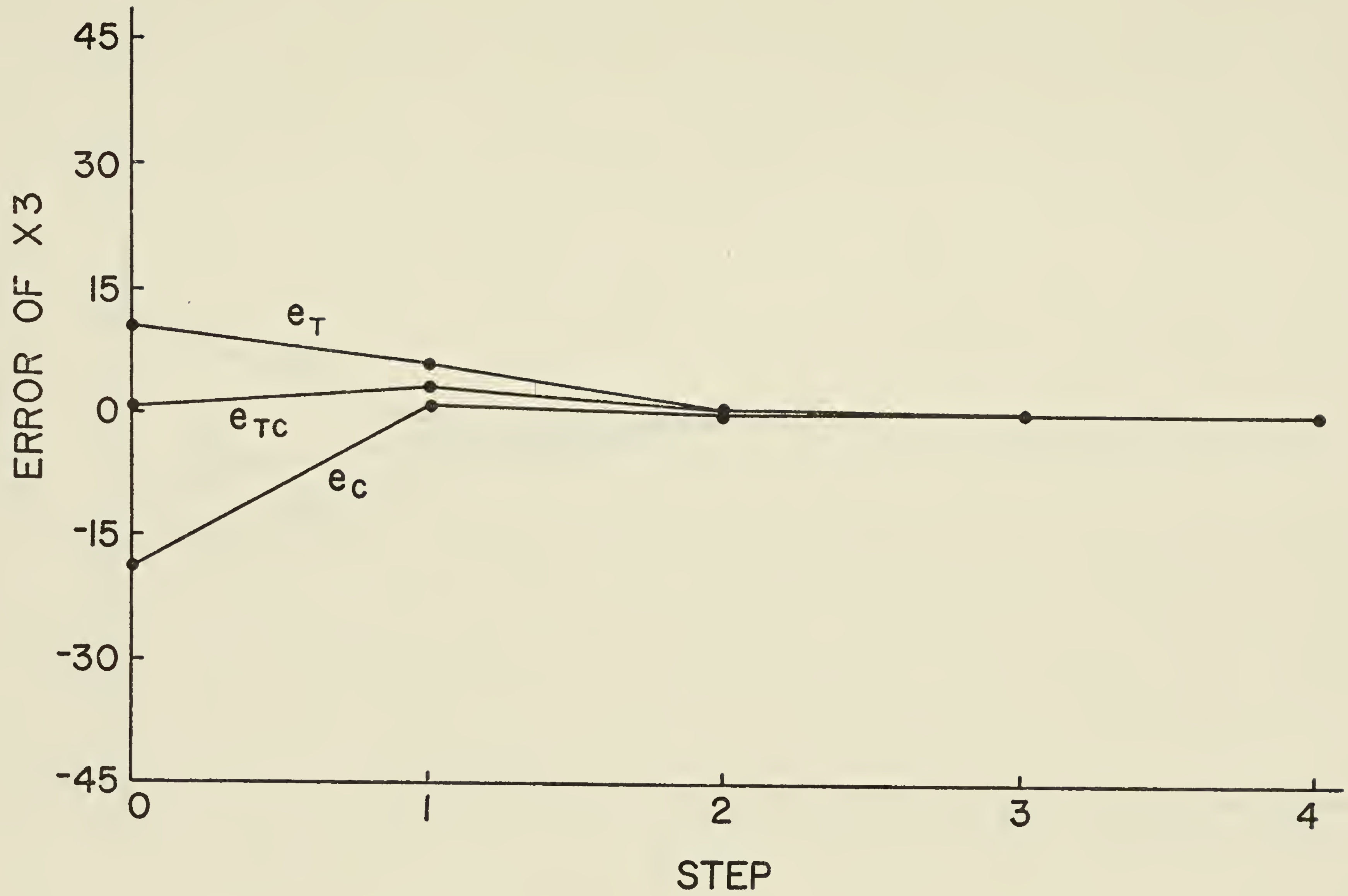

Figure 2(c): Open-Loop State Responses for $\hat{x}_3$

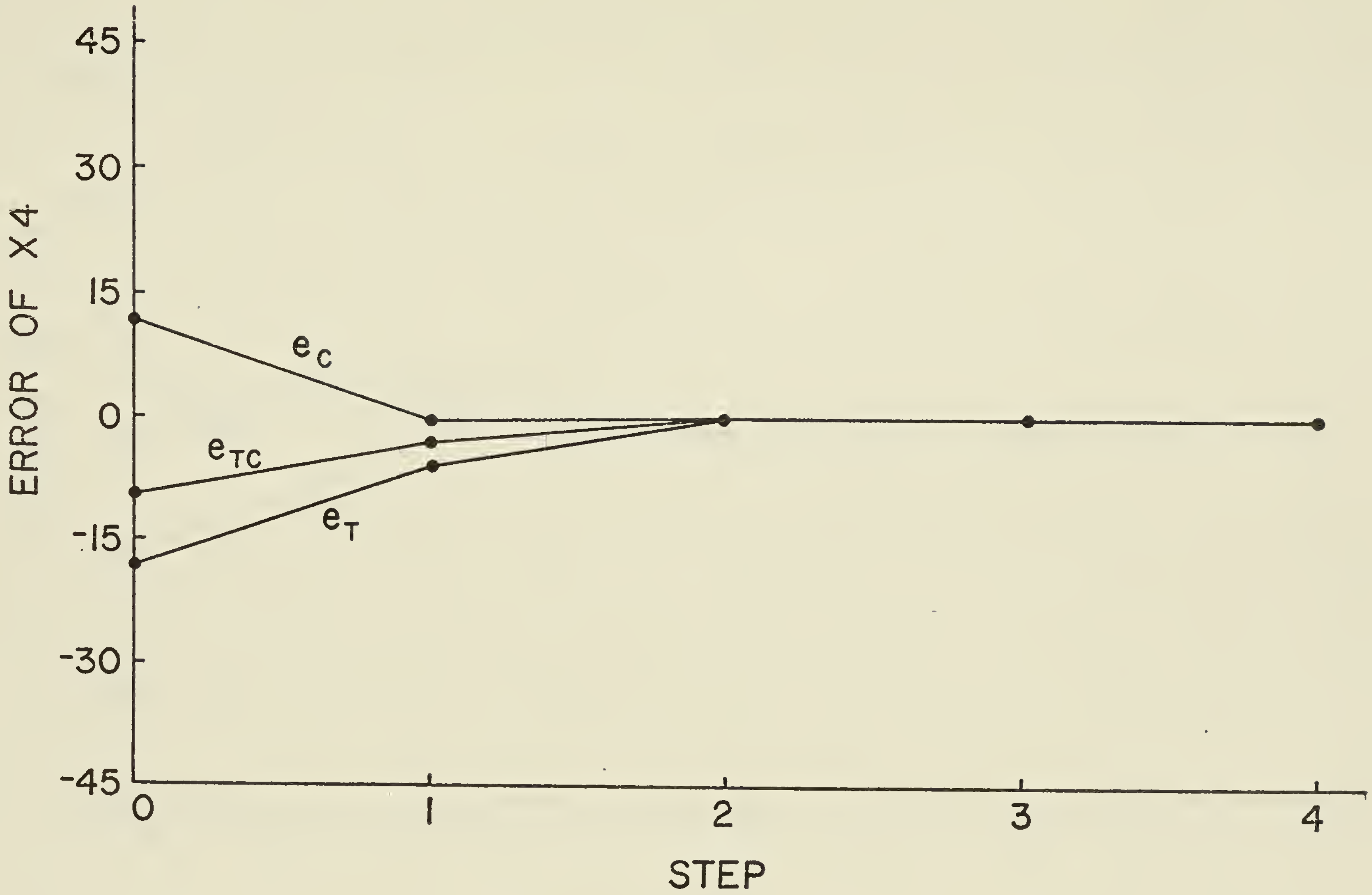

Figure 2(d): Open-Loop State Responses for $\hat{x}_4$

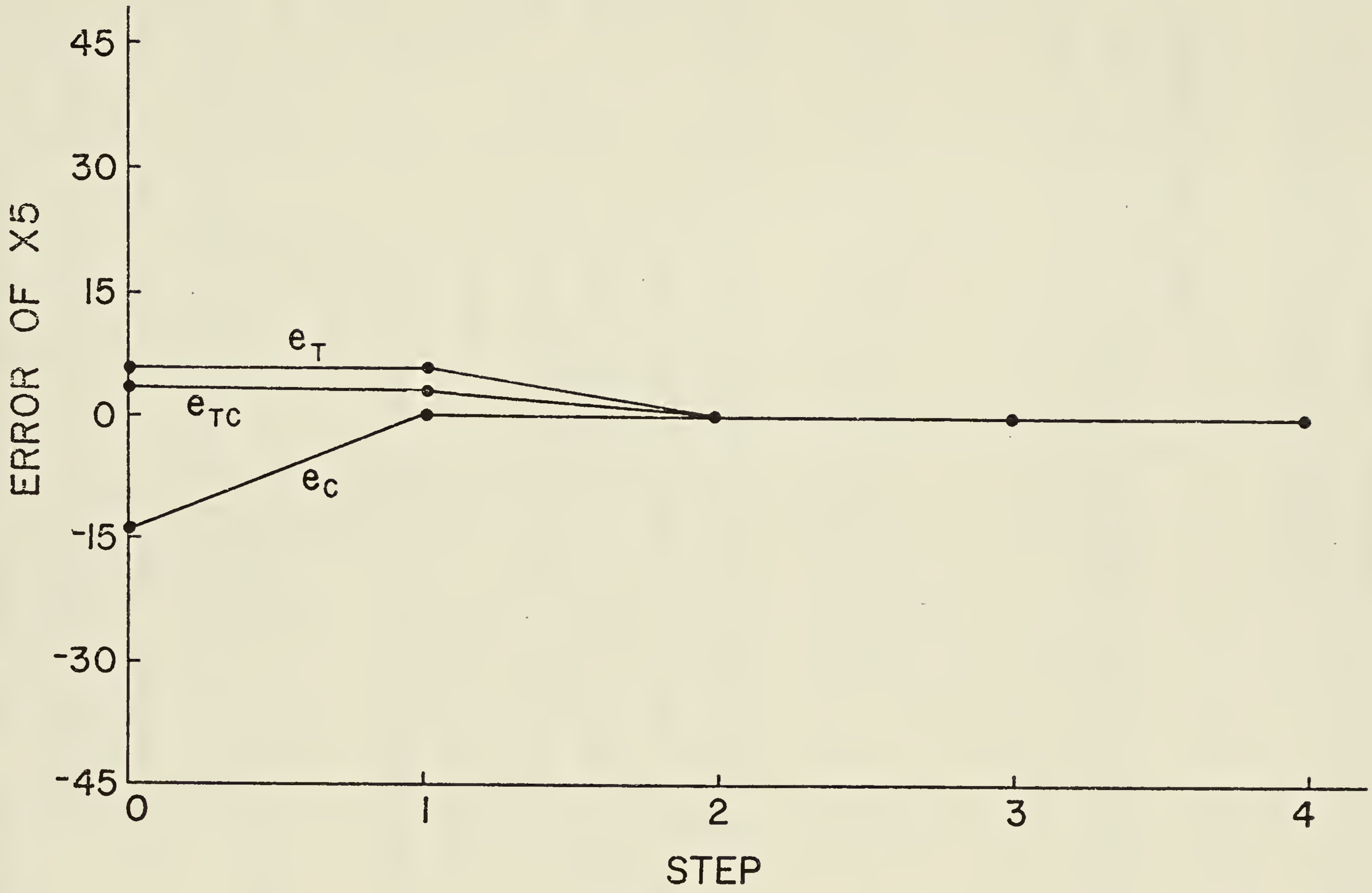

Figure 2(e): Open-Loop State Responses for $\hat{x}_5$

transient responses with Observers T and TC, the estimate of Observer TC traces more faithfully the actual state of the plant, although the trajectories of the responses are much similar.

5.3 Closed-loop Test

The controller which minimizes the performance index with complete state feedback is determined [14] to be

$$K = [2.04,\ 1.22,\ 4.09,\ 3.28,\ -0.22]$$

The same values of the initial states for the observers are set in the closed-loop system as in the open-loop system. Fig. 3 shows the errors in the closed-loop response. The entire states are shown in Appendix C.

Here, it is possible to note the similarities and differences in the behaviors of the closed-loop systems and the open-loop systems.

1) The steps required for convergence close enough to the actual state are almost same for all the observers in the closed-loop systems. However the error response of Observer C has large overshoot.

2) Due to introduction of the minimum-cost regulator, the estimation errors do not vanish in finite steps in

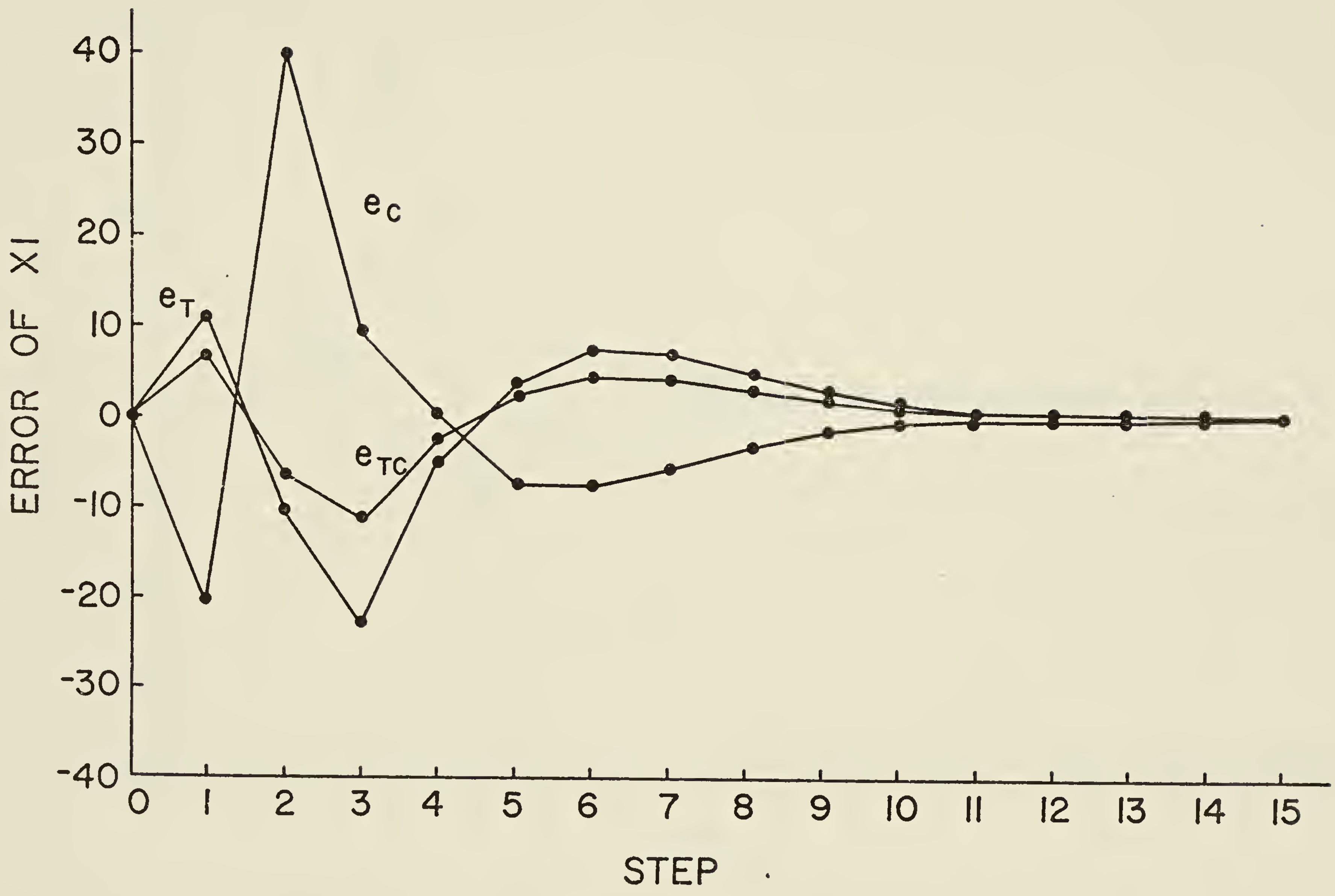

Figure 3(a): Closed-Loop State Responses for $\hat{x}_1$

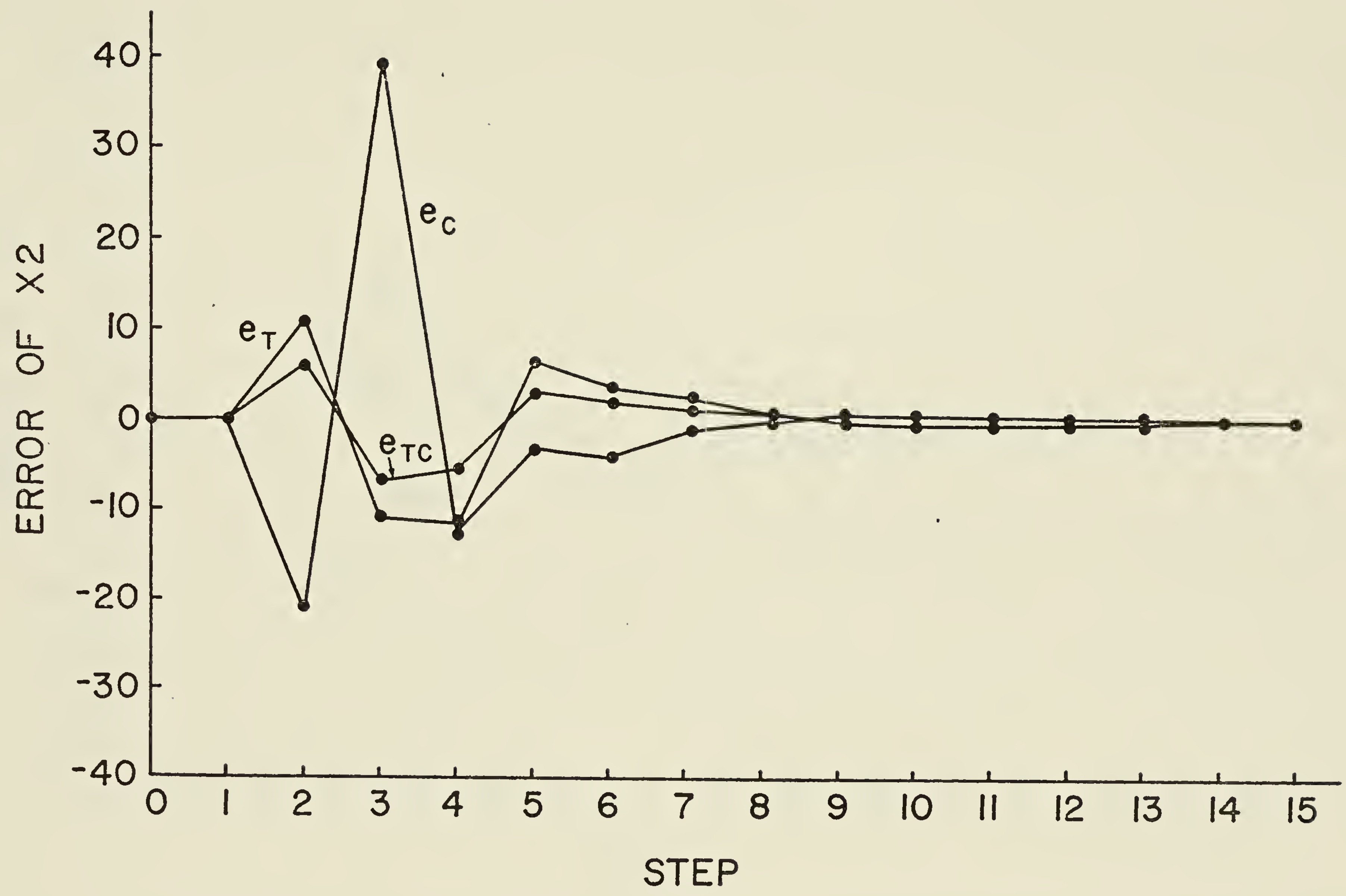

Figure 3(b): Closed-Loop State Responses for $\hat{x}_2$

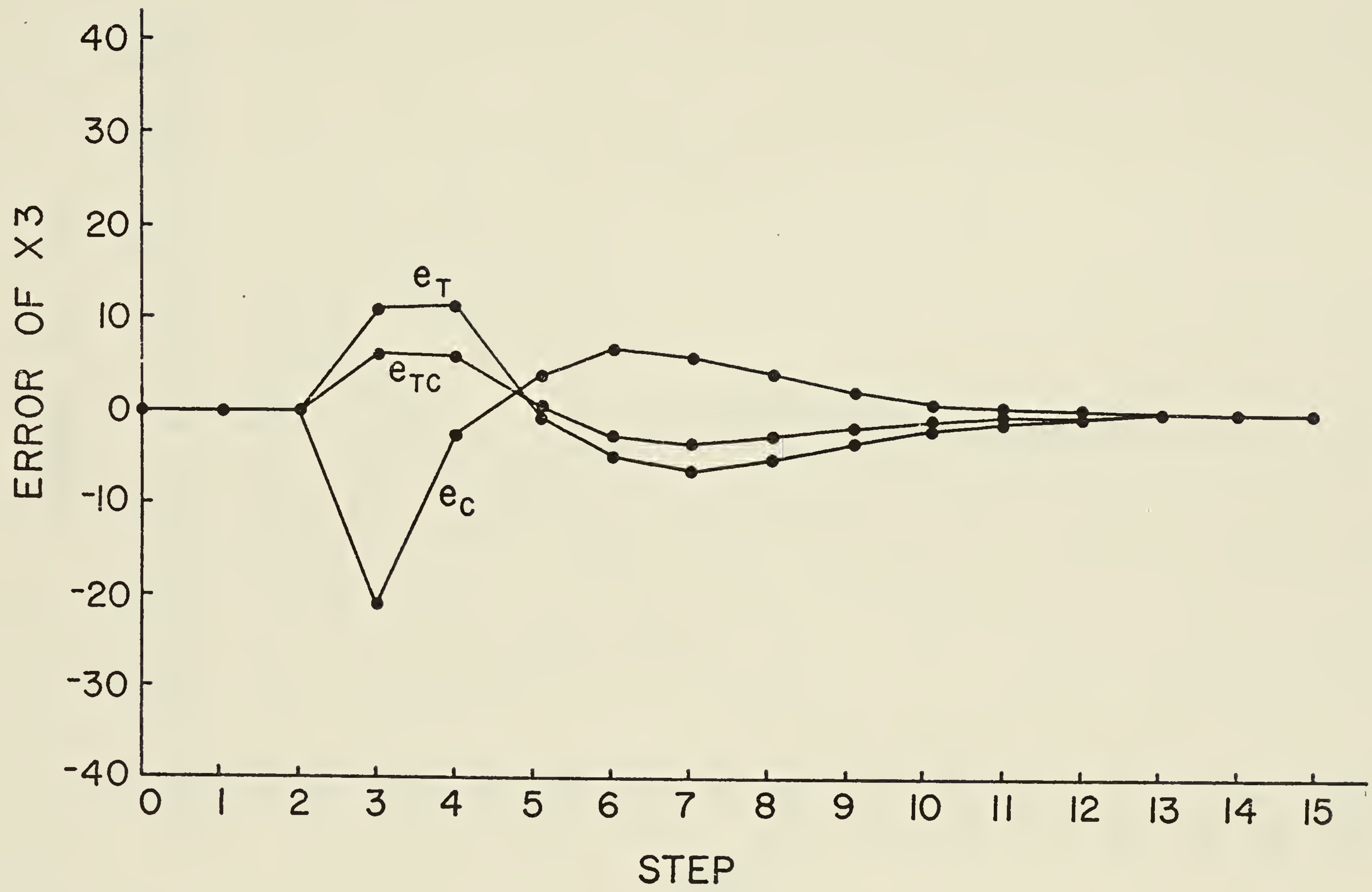

Figure 3(c): Closed-Loop State Responses for $\hat{x}_3$

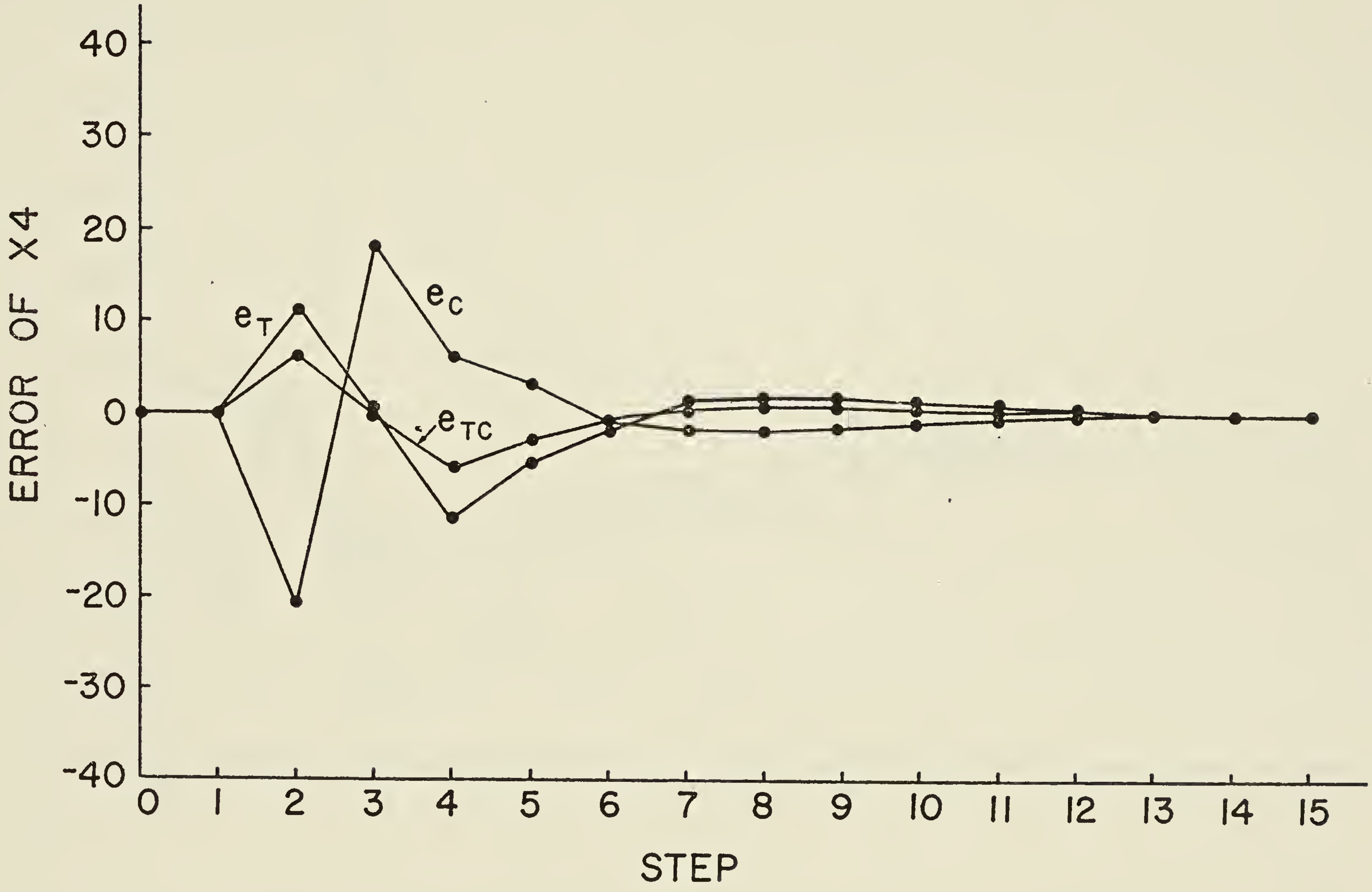

Figure 3(d): Closed-Loop State Responses for $\hat{x}_4$

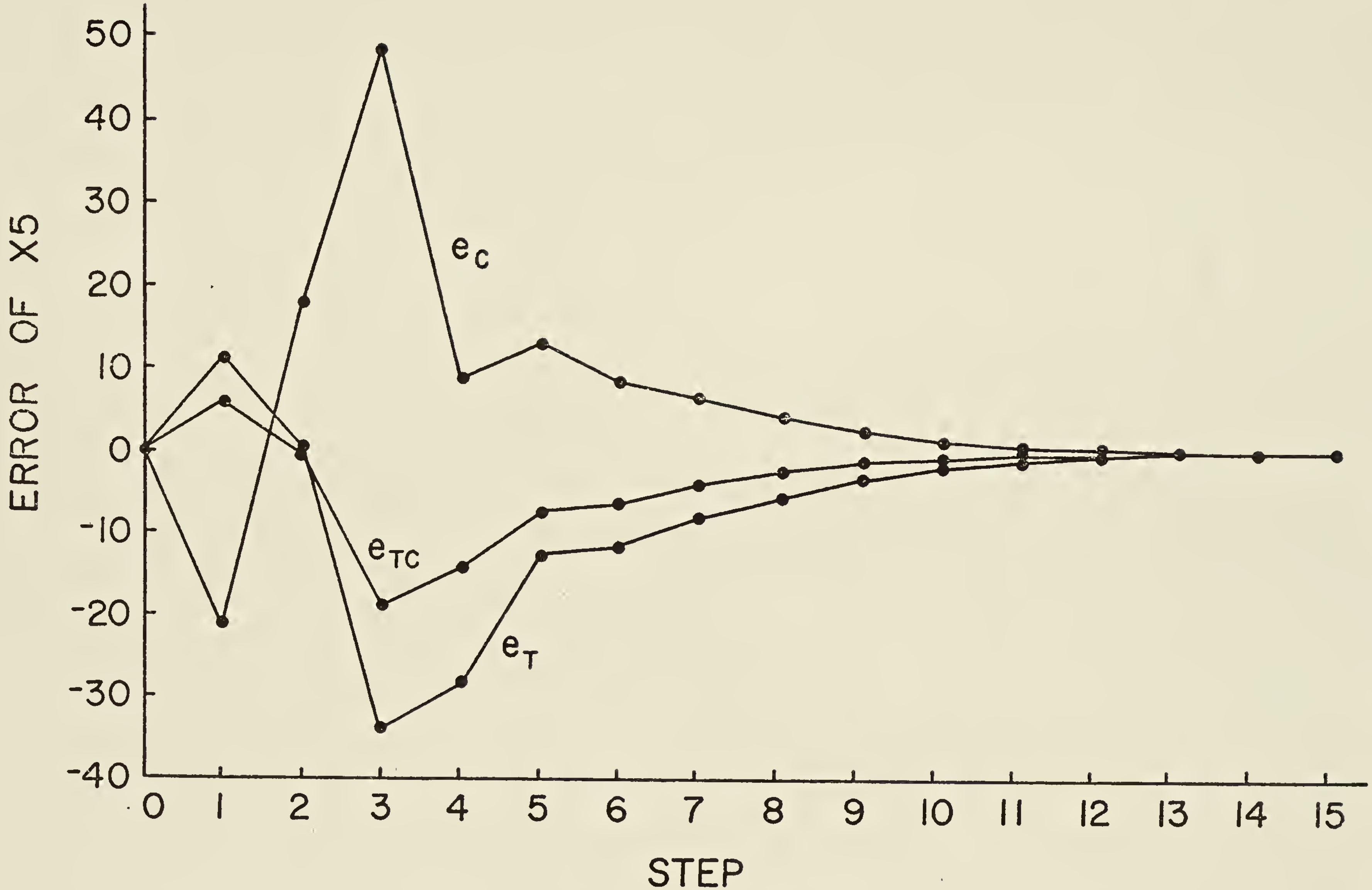

Figure 3(e): Closed-Loop State Responses for $\hat{x}_5$

the closed-loop systems for Observers T and TC as in the open-loop systems.

3) The behavior of the closed-loop system for Observer TC is very similar to that of Observer T, though the system with Observer T has a greater amount of overshoot than the system with Observer TC.

CHAPTER 6

CONCLUSIONS

6.1 Summary and Discussions of Results

The main work done in this thesis can be stated as; 1) discussing the various design methods developed so far for optimal discrete observers, by which the resulting observers minimize either of the minimum time or minimum cost criteria.

2) extending the above methods, resulting in a simple method which can be used to build an observer minimizing both of the criteria at the same time.

Design of a minimum-time observer is based in principle on a choice of a nilpotentent matrix with the minimum value of the nilpotency index, satisfying the observer equation. One method involves computing the projection of the n dimensional space of the plant observability matrix onto the n-m null space of the matrix C. This projection is then used to determine a suitable transformation linearly independent of the matrix C, which leads to a reduced order minimum-time observer. The minimum number of steps required to estimate an errorless state of the plant system is here uniquely determined by the observation characteristic of the plant, expressed in terms of the observability index.

While the other method has been developed through the synthesis of the plant in the Luenberger canonical representation, resulting in the simplicity of the structure of the observer, a particular choice of the transformation leads to the same result. This shows, in turn, the Luenberger canonical representation holds in its simple structure all the necessary information of the observation characteristics of the plant system, so that it could facilitate to design a minimum-time, minimum-cost observer.

The minimum-cost observer design problem is somewhat straightforward in comparison to the minimum-time observer design problem. The basic idea involved in solving this problem is to combine the preselected cost and estimation error equations and then convert this into an unconstrained minimization problem. The main difficulty in this procedure lies in the explicit dependence upon the initial condition on the plant state. This can be overcome simply by introducing an optimization in an average sense, so that the dependence is completely eliminated.

The elimination of the dependence upon the initial plant condition also indicates the independence of the choice of a transformation for the plant variables. Thus this enables the choice of a particular transformation so that the whole design procedure is considerably

simplified, which involves merely selecting the submatrices of the plant matrix in a canonical representation. This resulting observer appears to possess two properties, namely the number of steps required to obtain an errorless estimate of the plant state is minimized and also the cost in an average sense is minimized. In other words it is a minimum-time, minimum-cost observer.

While a simple example has been considered to demonstrate the superiority of the minimum-time, minimum-cost observer to the other observers discussed in this thesis, more extensive simulation and comparative studies need to be carried out with practical systems to assess the universal usefulness of this observer. It must be realized that minimization of cost in an average sense is not the same as minimization of the actual cost, but merely a bound on the worst case performance of the observer.

6.2 Suggestion for Further Research

The following is a partial list of possible areas for further work on the subject of observer:

1) Presence of plant parameter variation.

The work done in this thesis has been carried out with an assumption that the whole system is completely under fixed-parameter environment. However, in practice,

the parameters of the plant are subject to variation due to several factors such as, for example, change in environment and component ageing. Since the poles of the optimal observer are located at the origin on the complex frequency plane, perturbation due to the variation would not deteriorate the performance of the observer considerably. However, this could delay the convergence of the estimation error to zero. This perhaps requires modification of the transformation so as to reduce the effect of the variation on the observer performance, which has to be resolved before the design method discussed in this thesis can be used to practical plants.

2) Reduction in the dimension of an observer in control system.

In this thesis, the dimension of observers is uniquely determined by the number of the outputs of the plant. However, in the case that an observer is incorporated into a controller, the reduction in the dimension of an observer is possible, depending upon the controllability and observability of the plant, without any degradation of the overall performance. An attempt to solve this problem in a particular case has been reported in [10].

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APPENDIX A

Theorem

The complete observability of the pair (A,C) implies that

$$\text{rank } (C', A'C', \ldots, A'^{n-1}C') = n \tag{A-1}$$

that is, considering a particular form of the matrix $C = [I, 0]$,

$$\text{rank} \begin{bmatrix} I & 0 \\ A_{11} & A_{12} \\ A_{11}^2 + A_{12}A_{21}, & A_{11}A_{12} + A_{12}A_{22} \\ \vdots & \vdots \end{bmatrix} = n \tag{A-2}$$

Proof

Since the rank of matrices is not changed by adding to any row linear combination of other rows, we can write (A-2) as

$$\text{rank} \begin{bmatrix} I & 0 \\ --- & A_{12} \\ --- & A_{12}A_{22} \\ \vdots & \vdots \\ --- & A_{12}A_{22}^{n-1} \end{bmatrix} = n \tag{A-3}$$

Thus the matrix in eqn. (A-3) implies

$$\text{rank} \begin{bmatrix} 0 \\ A_2 \\ A_{12}A_{22} \\ \vdots \\ A_{12}A_{22}^{n-1} \end{bmatrix} = n-m \tag{A-4}$$

Since the dimension of A_{22} is n-m, by the Cayley-Hamilton theorem we need only include terms up to $A_{12}A_{22}^{n-m-1}$, which gives

$$\text{rank } [A_{12}', A_{22}'A_{12}', \ldots, A_{22}'^{n-m-1}A_{12}'] = n-m$$

we complete the proof.

APPENDIX B

We first assume that eqn. (2-6) holds. Then it immediately follows that

$$
\begin{array}{llllll}
Nx_1 = 0 & Nx_{p+1} = 0 & --- & Nx_{n-s+1} = 0 & \\
Nx_2 = x_1 & Nx_{p+2} = x_{p+1} & --- & Nx_{n-s+2} = x_{n-s+1} & \\
\vdots & \vdots & & \vdots & \qquad (B\text{-}1) \\
Nx_{p-1} = x_{p-2} & Nx_{p+q-1} = x_{p+q-2} & --- & Nx_{n-1} = x_{n-2} & \\
Nx_p = x_{p-1} & Nx_{p+q} = x_{p+q-1} & --- & Nx_n = x_{n-1} &
\end{array}
$$

where x_i $(i = 1,2,...,n)$ denotes the i-th column vector of T. For the sake of simplicity we also assume that

$$p \geqslant q \geqslant .... \geqslant s \qquad (B\text{-}2)$$

We define matrixes X_i $(i = 1,2,...,p)$ such that

$$
\begin{array}{ll}
X_1 & = [x_1, x_{p+1}, x_{p+q+1}, ..., x_{n-s+1}] \\
X_2 & = [s_2, x_{p+2}, ...] \\
\vdots & \qquad \vdots \\
X_{p-1} & = [x_{p-1}, x_{p+q-1}, ...] \\
X_p & = [x_p, x_{p+q}, ...]
\end{array}
\qquad (B\text{-}3)
$$

If the matrix X_i $(i = 1,2,...,p)$ has ℓ_i columns the assumption (B-2) implies that

$$\ell_p \leqslant \ell_{p-1} \leqslant \cdots \leqslant \ell_1 \tag{B-4}$$

Thus the array (B-1) can be rewritten in the following manner.

$$\begin{aligned} NX_1 &= 0 \\ NX_2 &= Z_1 \\ &\vdots \\ NX_p &= Z_{p-1} \end{aligned} \tag{B-5}$$

where the matrix Z_i $(i = 1,2,\ldots,p-1)$ consists of the first ℓ_{i+1} columns of X_i.

Next, we shall prove that $[X_1,X_2,\ldots,X_k]$ $(k = 1,2,\ldots,p)$ constitutes a basis for the null space of N^k. Since T is nonsingular and the vectors in (B-3) are simply a rearrangement of those in (B-1), the columns of $[X_1,X_2,\ldots,X_p]$ span n-dimensional space so that a basis for the null space of N^k can be found by determining all vectors α_i $(i = 1,2,\ldots,p)$ such that

$$N^k\ [X_1\alpha_1 + \cdots + X_p\alpha_p] = 0 \tag{B-6}$$

that is

$$N^k\ [X_{k+1}\alpha_{k+1} + \cdots + X_p\alpha_p] = 0 \tag{B-7}$$

If we multiply eqn. (B-6) by N^{p-k-1} it is easily seen that

$$N^{p-1} X_p \alpha_p = 0 \tag{B-8}$$

Since the columns of $N^{p-1}X_p$ are part of X_1 the columns of $N^{p-1}X_p$ are linearly independent.
Therefore,

$$\alpha_p = 0 \tag{B-9}$$

Similarly, the successive multiplication of eqn. (B-6) by N^{p-k-2}, N^{p-k-3}, ..., N gives

$$\alpha_p = \alpha_{p-1} = \dots = \alpha_{k+1} = 0 \tag{B-10}$$

Hence the columns of $[X_1, X_2, \dots, X_k]$ span the null space of N^k.

This suggests a method for the construction of the array (B-5). Choose a matrix W_1

$$W_1 = [w_1, w_2, \dots, w_r] \tag{B-11}$$

such that W_1 constitutes a basis for the null space of N. it is clear that the vectors in W_1 are also in the null space of N^2. Extend these vectors to form a basis

for the null space of N^2:

$$[W_1, \ldots, W_r, W_{r+1}, \ldots, W_s] = [W_1 : W_2] \tag{B-12}$$

Suppose that $N(W_2\alpha) = 0$ for some nonzero α. This $W_2\alpha$ is also in the null space of N that is,

$$W_2\alpha - W_1\beta = 0 \tag{B-13}$$

Since from eqn. (B-12) W_1 and W_2 are linearly independent, α and β must be zero. This implies that the columns NW_2 are in the null space of N, being independent of W_1. Proceeding in this manner we can construct the array (B-5), which can be rewritten in the form of (B-1). Thus the assumption is correct.

APPENDIX 3

The state with complete feedback is referred to as "x".

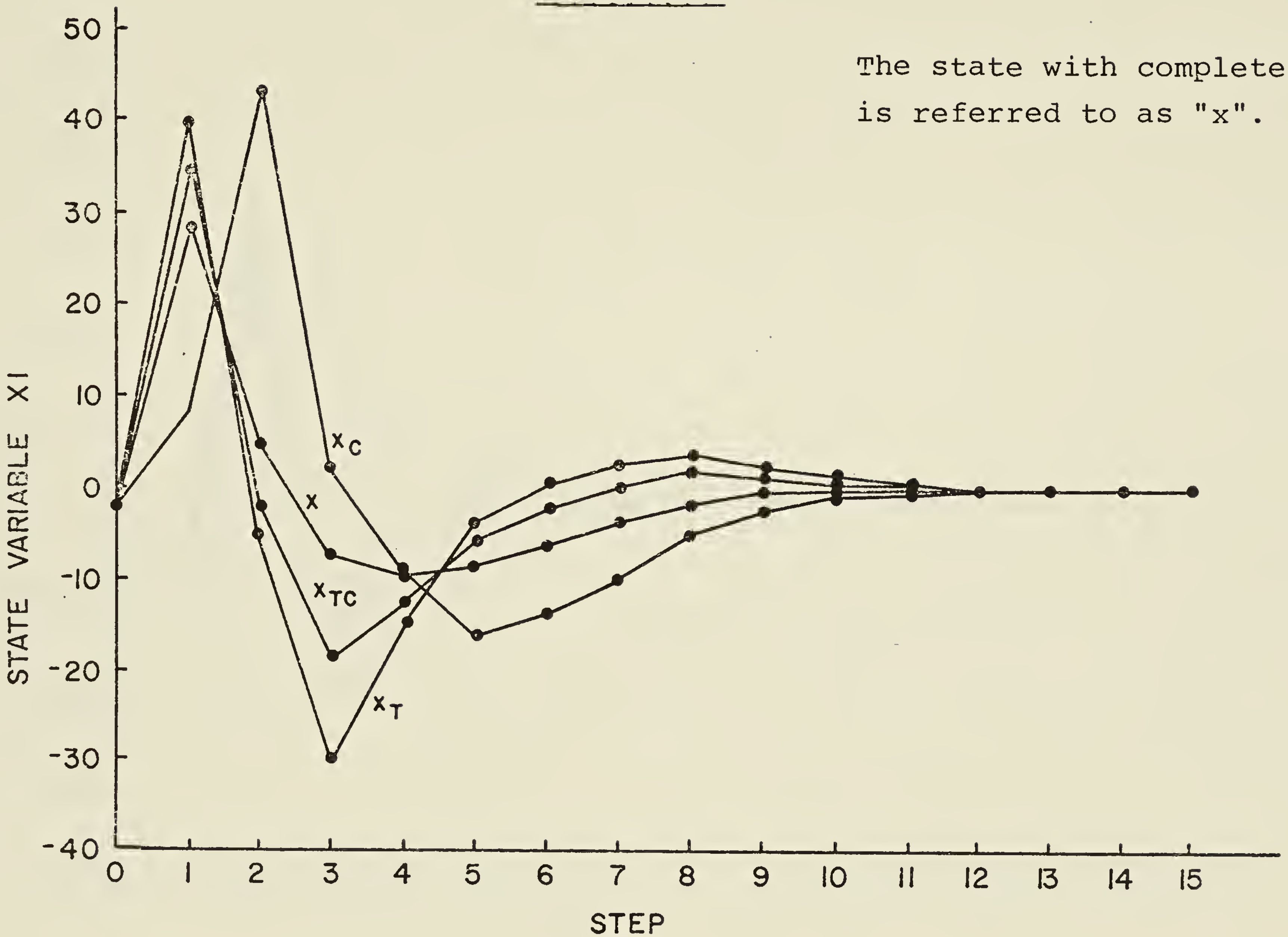

Figure 4(a): Plant State Responses of the Closed-Loop System for x_1

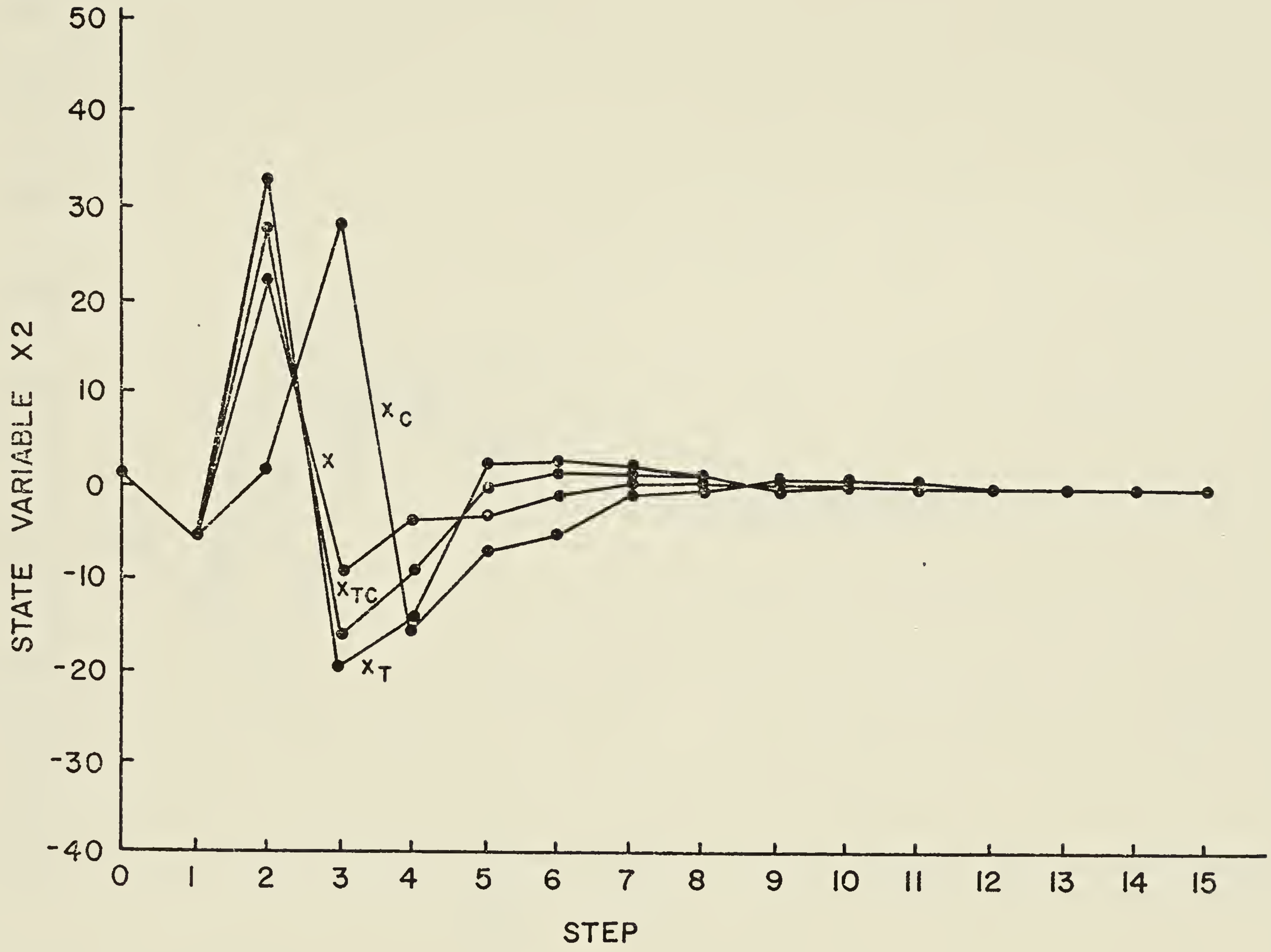

Figure 4(b): Plant State Responses of the Closed-Loop System for x_2

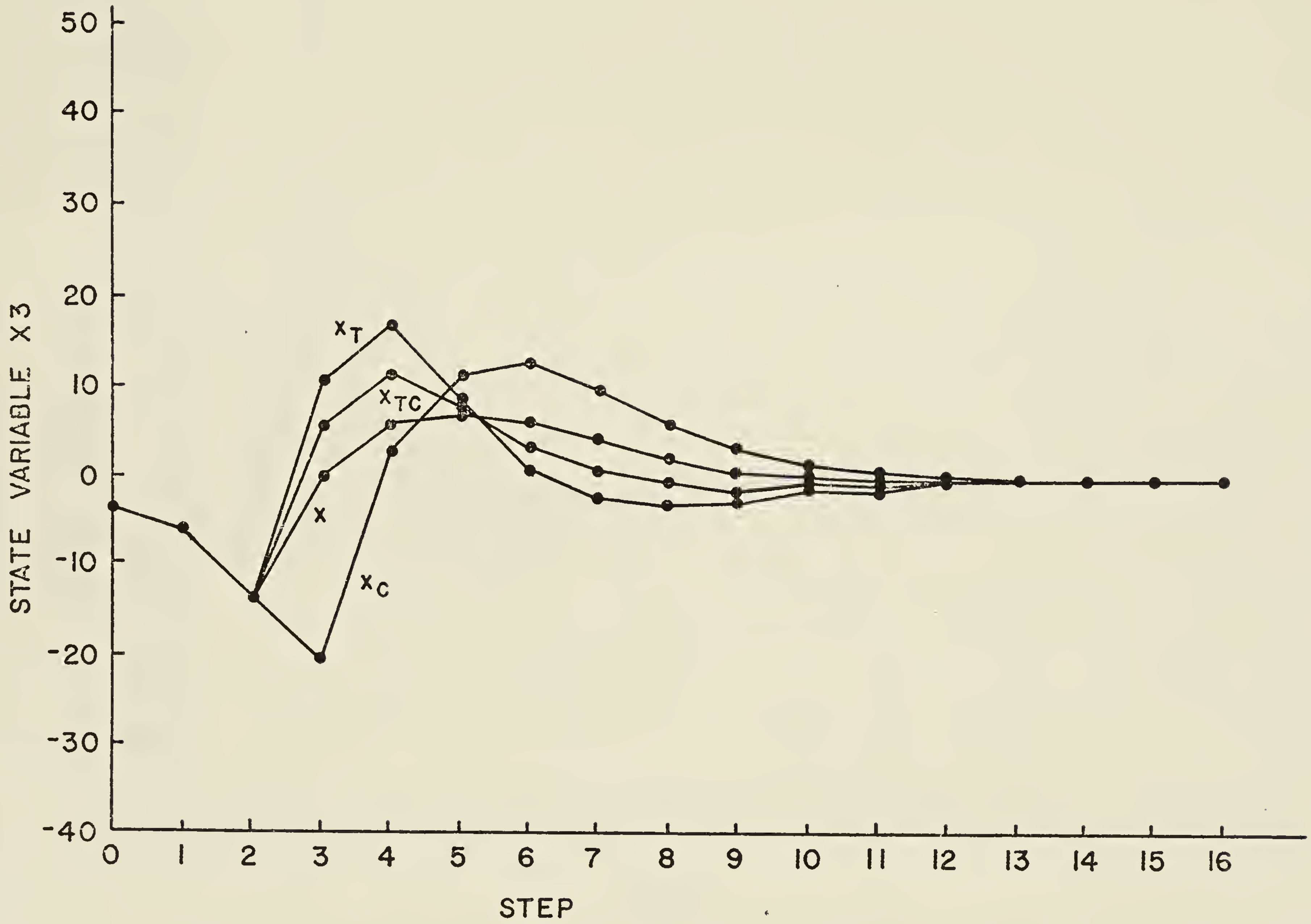

Figure 4(c): Plant State Responses of the Closed-Loop System for x_3

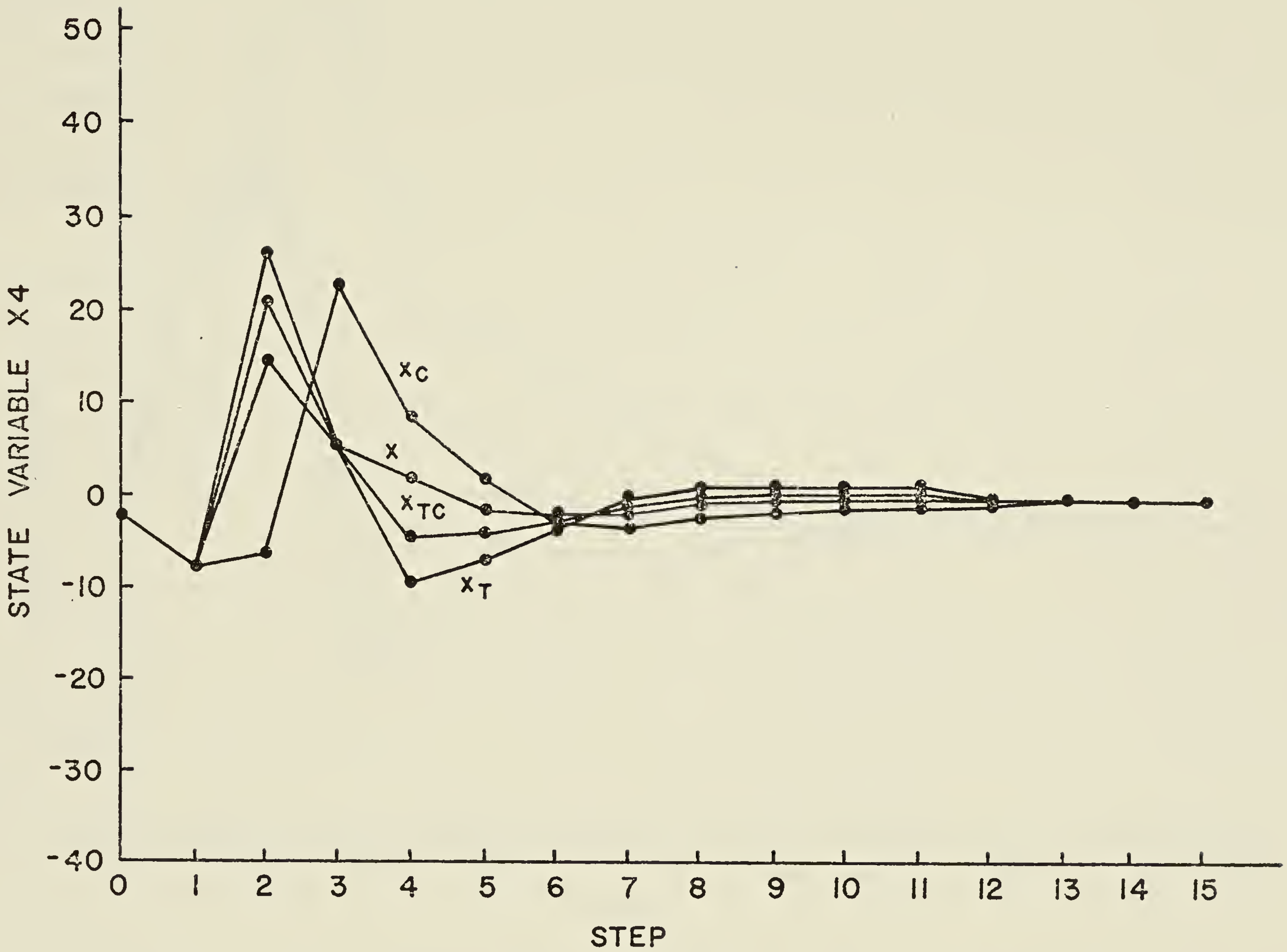

Figure 4(d): Plant State Responses of the Closed-Loop System for x_4

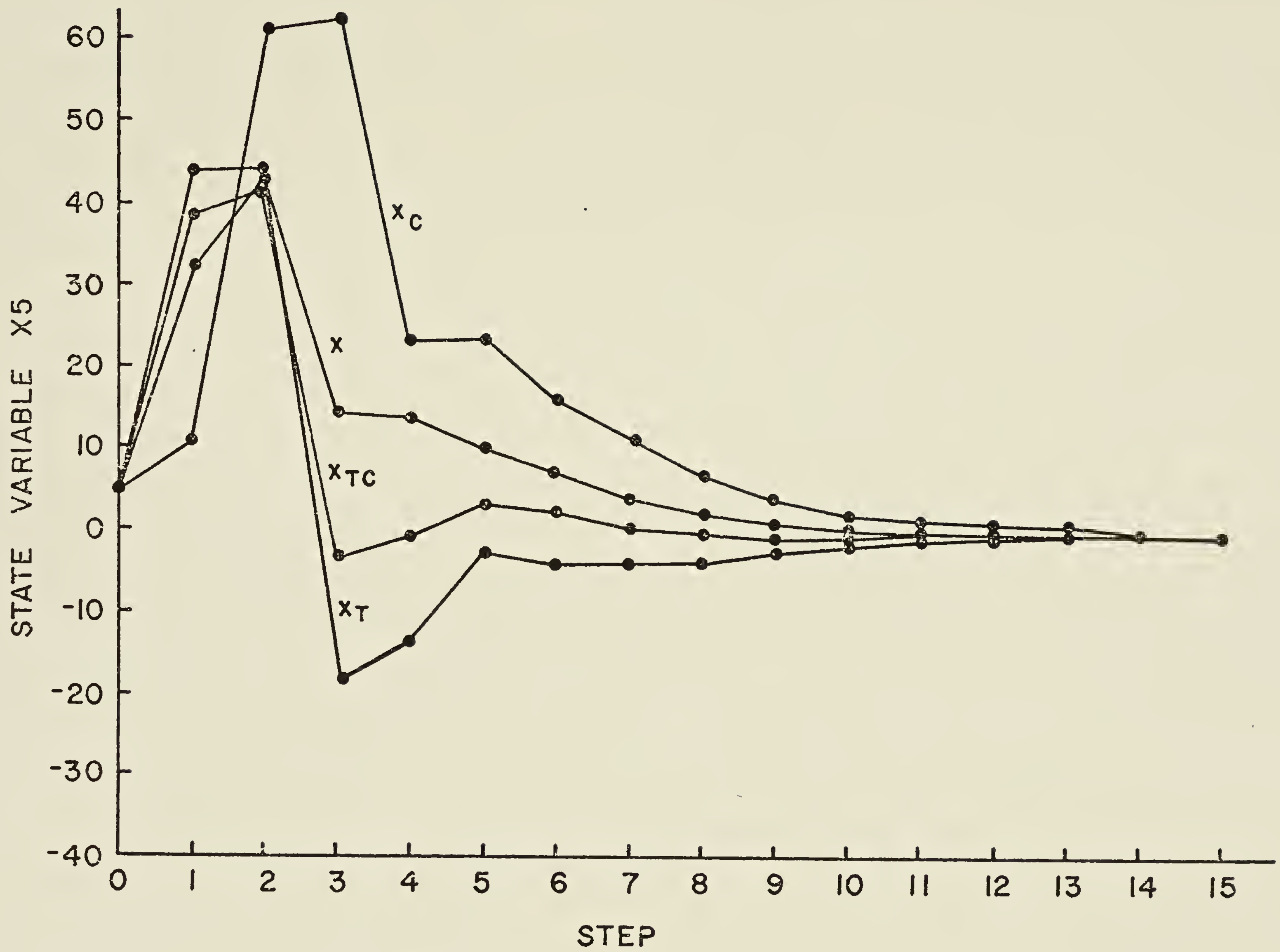

Figure 4(e): Plant State Responses of the Closed-Loop System for x_5

[illegible]